

THE POWER LAW FOR BUFFON'S NEEDLE LANDING NEAR THE SIERPINSKI GASKET

MATTHEW BOND AND ALEXANDER VOLBERG

ABSTRACT. In this paper we get a power estimate from above of the probability that Buffon's needle will land within distance 3^{-n} of Sierpinski's gasket of Hausdorff dimension 1. In comparison with the case of 1/4 corner Cantor set considered in Nazarov, Peres, and the second author [14]: we still need the technique of [14] for splitting the directions to good and bad ones, but the case of Sierpinski gasket is considerably more generic and lacks symmetry, resulting in a need for much more careful estimates of zeros of the Fourier transform of Cantor measure.

1. Introduction

Among self-similar planar sets of Hausdorff dimension 1, some of the simplest are the Sierpinski gasket \mathcal{G} (formed by three self-similarities by the scaling factor 1/3) and the square 4-corner Cantor set \mathcal{K} (formed by four self-similarities by the scaling factor 1/4; it is a cartesian product of two Cantor sets in \mathbb{R}). By the Besicovitch projection theorem [13], these irregular sets of finite Hausdorff H^1 measure must have zero length in almost every orthogonal projection onto a line. One may partially construct these sets in the usual way by taking their convex hulls and then taking the union of all possible images of n -fold compositions of the similarity maps, which we call \mathcal{G}_n and \mathcal{K}_n , respectively. Then $\mathcal{G} = \bigcap_n \mathcal{G}_n$ and $\mathcal{K} = \bigcap_n \mathcal{K}_n$. One may then ask the rate at which the Favard length – the average over all directions of the length of the orthogonal projection onto a line in that direction – of these sets \mathcal{G}_n and \mathcal{K}_n decay to zero as a function of n ⁽¹⁾. For bounded sets, Favard length

1991 *Mathematics Subject Classification*. Primary: 28A80. Fractals, Secondary: 28A75, Length, area, volume, other geometric measure theory 60D05, Geometric probability, stochastic geometry, random sets 28A78 Hausdorff and packing measures.

Research of the authors was supported in part by NSF grants DMS-0501067, 0758552 .

¹Such decay must occur by the Besicovitch projection theorem and by continuity of measures, since we're taking the Lebesgue measure of decreasing sets in the parameter space of $\{\text{directions}\} \times \{\text{projected x values}\}$.

is also called Buffon needle probability, since up to a normalization constant, it is the likelihood that a long needle dropped with independent, uniformly distributed orientation and distance from the origin will intersect the set somewhere. Observe that \mathcal{G}_n and \mathcal{K}_n are in some sense comparable to small neighborhoods of \mathcal{G} and \mathcal{K} , so that $Fav(\mathcal{G}_n)$ is comparable to the likelihood that “Buffon’s needle” will land in a 3^{-n} -neighborhood of \mathcal{G} .

The first quantitative results for the Favard length problem were obtained in [16],[18]; in the latter paper a general way of making a quantitative statement from the Besicovitch theorem is considered. But being rather general, this method does not give a good estimate for self-similar structures such as \mathcal{K}_n or \mathcal{G}_n .

Indeed, vastly improved estimates have been proven in these cases: in [14], it was shown that for $p < 1/6$, $Fav(\mathcal{K}_n) \leq \frac{c_p}{n^p}$, and the current paper extends this result to \mathcal{G}_n for some other $p > 0$. These results cannot possibly be improved to $p = 1$: $Fav(\mathcal{K}_n) \geq c \frac{\log n}{n}$ (This is [1]⁽²⁾, and the argument and result also apply to \mathcal{G}_n .) Compare this with [16], in which it was shown that certain random sets of which \mathcal{K}_n is a special case almost surely decay in Favard length like $\frac{1}{n}$.

Crucial to [1] was a tiling property: namely, under orthogonal projection on the line with slope $1/2$, the squares composing \mathcal{K}_n tile a line segment. Oddly enough, such a property can be used to prove upper bounds as well: under the assumption that some orthogonal projection in some direction contains an interval, Laba and Zhai [9] showed that the result of [14] holds for Cantor-like product sets of finite H^1 measure (but with a smaller exponent). Their argument uses tiling results obtained in Kenyon [8] and Lagarias-Wang [10] to fill in a gap where [14] fails to generalize (more on this shortly).

With the exception of [16] and [18], the above papers all extract their results from information about L^2 norms of the projection multiplicity function, which counts how many squares (or triangles) project to cover each point. The function $f_{n,\theta} : \mathbb{R} \rightarrow \mathbb{N}$ is defined by

$$f_{n,\theta} = \sum_{\text{Sierpinski triangles } T \text{ of } \mathcal{G}_n} \chi_{proj_\theta(T)}.$$

Note that $Fav(\mathcal{G}_n) = \pi^{-1} \int_0^\pi |supp(f_{n,\theta})| d\theta$. In [14] and [1], the L^2 norm of the analog of this function for squares was studied to obtain Buffon needle probability

²the method is stable under “bending the needle” slightly - see [5].

estimates for \mathcal{K}_n – in [1], $p = 1, 2$ were related to $\chi_{\text{supp}(f_{n,\theta})}$ via the Cauchy inequality, while in [14], $p = 2$ was studied via Fourier transforms and related to the measure of the level sets of f .

Consider some heuristics. Let $f : [0, 1] \rightarrow \mathbb{N}$ be any sum of measurable characteristic functions such that $\|f\|_{L^1} = 1$. If the mass is concentrated on a small set, the L^p norm should be large for $p > 1$. Thus a large L^p norm should indicate that the support of a function is small, and vice versa. Let $K > 0$, let $A = \text{supp}\{f\}$, and let $A_K = \{x : f \geq K\}$. $1 = \int f \leq \|f\|_p \|\chi_A\|_q$, so $m(A) \geq \|f\|_p^{-q}$, a decent estimate. The other basic estimate is not so sharp: $m(A) \leq 1 - (K - 1)m(A_K)$. However, a combinatorial self-similarity argument of [14] shows that for the Favard length problem, it bootstraps well under further iterations of the similarity maps – this argument is revisited in Section 3. Hence, up to some loss of sharpness, it has been shown that to study Favard length of these self-similar sets, it is necessary and sufficient to study the L^2 norms of $f_{n,\theta}$.⁽³⁾

One must average $|\text{supp}f_{n,\theta}|$ over the parameter θ to get Favard length of \mathcal{G}_n , and for some directions, the orthogonal projections do not even decay to length zero with n (i.e., the L^2 norms of $f_{n,\theta}$ are bounded for these angles), and this countable dense set of directions is to a large extent classified in [8]. In [14], a method for controlling the measure of a set of angles E on which the projections fail to decay rapidly was found: one takes the Fourier transform of $f_{n,\theta}$ in the length variable, and takes a sample integral of $|\hat{f}_{n,\theta}(x)|^2$ over a chosen small interval I where $\int_{E \times I} |\hat{f}_{n,\theta}(x)|^2 d\theta dx$ is small. One then shows that there is a $\theta \in E$ such that $\int_I |\hat{f}_{n,\theta}(x)|^2 dx$ is not too small relative to the size of E , and so E must be small.

In all cases, $\hat{f}_{n,\theta}$ is a self-similar product $\prod_k \varphi_\theta(3^{-k}y)$ of trigonometric polynomials φ_θ . The danger is that the low-frequency zeroes might kill off the better-behaved high-frequency terms. In [14], the four frequencies of φ_θ were symmetric around 0, allowing the terms to simplify to two cosines, and trig identities allowed the whole product to be estimated by a single sine term. In [9], an analogous role was played by tiling, and the product structure allowed for a change and separation of variables. In the current case, \mathcal{G}_n , neither of these things happen, but our considerations show that a so-called “analytic tiling” on the Fourier side (Section 5) proves that the complex zeroes from different factors are separated away from each other,

³So far, only L^p for $p = 1, 2$, or ∞ have played any useful role, to our knowledge.

preventing any resonance that may have caused the set of small values to grow too large.

Separating variables is more difficult when there is no product structure, so instead we isolated the zeroes in small intervals and found estimates valid for each small interval around each zero, so that an estimate on medium-frequency terms could be made independent of x . These zeroes λ_j depend on θ , so we traced how the zeroes λ_j of φ move as θ varies. In order for our estimates to work, we needed the real parts of the λ_j to move at a more or less constant rate without too many oscillations, allowing the path integral of the Riesz product to be controlled by the basic integral of the Riesz product on $[0, 2\pi]$. These technicalities are resolved in Section 4. To get such highly regular behavior in the zeroes λ_j as functions of θ , we had to consider them as functions of a complex variable $\zeta = \theta + i\sigma$ and appeal to holomorphic function theory.

The case of the gasket is much closer to the generic self-similar case as φ becomes a rather general 3-term exponential sum, providing a much better glimpse at the general Besicovitch irregular self-similar set than the sets considered in [14] and [9]. We believe that using this approach one can work with all such sets. However, there are a couple of problems which remain unresolved for now: see Section 8.

Rather strangely, a claim in the spirit of the Carleson Embedding Theorem, in the form of Lemma 27, plays an important part in our reasoning. Because the Fourier transform turns stacks of triangles (i.e., sums of overlapping characteristic functions) into clusters of frequencies, this lemma provides important upper bounds when θ belongs to E .

The main result of this article is the following estimate.

Theorem 1. *For each $p < 1/14$, there exists $C_p > 0$ such that for all $n \in \mathbb{N}$,*

$$Fav(\mathcal{G}_n) \leq \frac{C_p}{n^p}.$$

One can do better than this exponent, but we chose to simplify the argument. The changes needed amount to nothing more than some $1 + \epsilon$ bookkeeping. The sharper exponent $p < 1/8$ is given in Section 8, in which degenerate gaskets are also considered.

2. THE FOURIER-ANALYTIC PART

2.1. The setup. The goal of this section is to prove Theorem 3, which shows that for most directions, a considerable amount of stacking occurs when the triangles are projected down. Throughout the paper, the constants c and C will vary from line to line, but will be absolute constants not depending on anything. The symbols c and C will typically denote constants that are sufficiently small or large, respectively. Everywhere we use the definition $B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$.

For convenience, we will now rescale \mathcal{G}_n by a factor absolutely comparable to 1 and bound the triangles by discs and study this set instead. That is, for $\alpha \in \{-1, 0, 1\}^{n+1}$ let

$$z_\alpha := \sum_{k=0}^n \left(\frac{1}{3}\right)^k e^{i\pi[\frac{1}{2} + \frac{2}{3}\alpha_k]},$$

and then let

$$\mathcal{G}_n := \bigcup_{\alpha \in \{-1, 0, 1\}^{n+1}} B(z_\alpha, 3^{-n}).$$

Note that \mathcal{G}_n has 3^{n+1} discs of radius 3^{-n} . After a rescaling, the usual $n + 1$ st Sierpinski gasket (composed of 3^{n+1} triangles) sits inside of \mathcal{G}_n . We may still speak of the approximating discs as ‘‘Sierpinski triangles.’’

Observe that $f_{n,\theta} = \nu_n * 3^n \chi_{[-3^{-n}, 3^{-n}]}$, where $\nu_n := *_{k=0}^n \tilde{\nu}_k$ and

$$\tilde{\nu}_k = \frac{1}{3} [\delta_{3^{-k} \cos(\pi/2 - \theta)} + \delta_{3^{-k} \cos(-\pi/6 - \theta)} + \delta_{3^{-k} \cos(7\pi/6 - \theta)}].$$

We will now slightly modify f for convenience. Note that

$$\hat{f}_{n,\theta}(x) = 3^n \hat{\chi}_{[-3^{-n}, 3^{-n}]}(x) \cdot \prod_{k=0}^n \phi_\theta(3^{-k}x),$$

where $\phi_\theta(x) = \frac{1}{3} [e^{-i \cos(\theta - \pi/2)x} + e^{-i \cos(\theta - 7\pi/6)x} + e^{-i \cos(\theta + \pi/6)x}]$. By factoring and changing the variable, we may instead write in place of ϕ_θ the function

$$\varphi_t(x) = \frac{1}{3} [1 + e^{-itx} + e^{-ix}], \quad t \in [0, 1] \quad (2.1)$$

To do this, we split $[0, 2\pi]$ into six cases: consider \mathcal{G}_0 , which has three triangle centers. Under the projection map, a middle point migrates between the other two, either forward or in reverse. The change from θ to the parameterization t corresponds to translating and rescaling the projections so that two projected triangles on the ends remain stationary during this migration of the middle triangle. In particular, we abolish θ and write $f_{n,t}$ from now on. We allow ourselves to drop

the t from functions and sets that depend on it when this dependence is not the pertinent feature in an argument.

For numbers $K, N > 0$, define the following, (also depending on t where appropriate):

$$f_N^*(s) := \sup_{n \leq N} f_{n,t}(s) \quad (2.2)$$

$$A_K^* := \{s : f_N^*(s) \geq K\} \quad (2.3)$$

$$E := \{t : |A_K^*| \leq \frac{1}{K^3}\}. \quad (2.4)$$

E is essentially the set of pathological t such that $\|f_{n,t}\|_{L^2(s)}$ is small for all $n \leq N$, as in [14]. In fact, we have this result, proved in Section 7:

Theorem 2. *Let $t \in E$. Then*

$$\max_{n: 0 \leq n \leq N} \|f_{n,t}\|_{L^2(s)}^2 \leq cK.$$

The aim of Section 2 is to prove the following:

Theorem 3. *Let $\epsilon_0 < 1/11$. Then for $N \gg 1$, $|E| < N^{-\epsilon_0}$.*

So let $K \approx N^{\epsilon_0}$, and suppose $|E| > \frac{1}{K}$. We will show that $N < N^*$, for some finite constant $N^* \gg 1$.

2.2. Initial reductions. Because of Theorem 2, we have $\forall t \in E$,

$$K \geq \|f_{N,t}\|_{L^2(s)}^2 \approx \|\widehat{f_{N,t}}\|_{L^2(x)}^2 \geq C \int_1^{3^{N/2}} |\widehat{\nu_N}(x)|^2 dx \quad (2.5)$$

Let $m \approx 2\epsilon_0 \log N \approx 2 \log K$. Split $[1, 3^{N/2}]$ into $N/2$ pieces $[3^k, 3^{k+1}]$ and take a sample integral of $|\widehat{\nu_N}|^2$ on a small block $I := [3^{n-m}, 3^n]$, with $n \in [N/4, N/2]$ chosen so that

$$\frac{1}{|E|} \int_E \int_{3^{n-m}}^{3^n} |\widehat{\nu_N}(x)|^2 dx dt \leq CKm/N.$$

This choice is possible by (2.5). Define

$$\tilde{E} := \{t \in E \setminus [1/2 - 3^{-m}, 1/2 + 3^{-m}] : \int_{3^{n-m}}^{3^n} |\widehat{\nu_N}(x)|^2 dx dt \leq 2CKm/N\}.$$

It then follows that $|\tilde{E}| \geq \frac{1}{2K}$. We removed a small interval (of size $1/K^2$) around $1/2$ so that we may freely assume $|\varphi'_t(\lambda_j)| > c3^{-m}$ for all complex zeroes λ_j of φ having small enough imaginary part. It is an elementary consideration, but see also Section 4.

Note that $\widehat{\nu}_N(x) = \prod_{k=0}^N \varphi(3^{-k}x) \approx \prod_{k=0}^n \varphi(3^{-k}x)$ for $x \in [3^{n-m}, 3^n]$.

So for $t \in E$,

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^n |\varphi_t(3^{-k}x)|^2 dy d\theta \leq \frac{CKm}{N} \leq 2\epsilon_0 N^{\epsilon_0-1} \log N.$$

Later, we will show that $\exists t \in E$ such that

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^n |\varphi_t(3^{-k}x)|^2 dx \geq c3^{m-2\cdot 3m} = c3^{-5m} = cN^{-10\epsilon_0}. \quad (2.6)$$

The result: $2\epsilon_0 \log N \geq N^{1-11\epsilon_0}$, i.e., $N \leq N^*$. In other words:

Proposition 4. *Inequality (2.6) is sufficient to prove Theorem (3). Further, inequality 2.6 can be deduced from Propositions 5 and 6, as will be seen shortly.*

So let us prove inequality (2.6).

First, let us write $\prod_{k=0}^n \varphi_t(3^{-k}x) = P_t(x) = P_{1,t}(x)P_{2,t}(y)$, where P_2 is the low frequency part, and P_1 is has medium and high frequencies:

$$P_{1,t}(x) := \prod_{k=0}^{n-m} \varphi_t(3^{-k}x) = \widehat{\nu}_{n-m}(x)$$

$$P_{2,t}(x) = \prod_{k=n-m+1}^n \varphi_t(3^{-k}x) = \widehat{\nu}_{m-1}(3^{m-1-n}x)$$

We want the following:

Proposition 5. *Let $t \in E$ be fixed. Then $\int_{3^{n-m}}^{3^n} |P_{1,t}(x)|^2 dx \geq C3^m$.*

We also want a proportion of the contribution to the integral separated away from the complex zeroes of $P_{2,t}$:

Proposition 6. *Let $SSV(t) := \{x \in I : |P_{2,t}(x)| \leq 3^{-3m}\}$. Then*

$$\frac{1}{|\tilde{E}|} \int_{\tilde{E}} \int_{SSV(t)} |P_{1,t}(x)|^2 dx dt \leq c3^m,$$

where c is less than the C from Proposition 5.

$SSV(t)$ is so named because it is the **set of small values** of P_2 on I . Note that while Proposition 5 will be proven for all $t \in \tilde{E}$, Proposition 6 is an average. But from the average, one will be able to extract some $t \in \tilde{E}$ so that

$$\int_{SSV(t)} |P_{1,t}(x)|^2 dx \leq c3^m,$$

and so combining this with Proposition 5,

$$\int_{I \setminus SSV(t)} |P_{1,t}(x)|^2 dx \geq c' 3^m,$$

Thus Propositions 5 and 6 suffice to prove Theorem 3, and Proposition 4 has been demonstrated.

Also, one may recall that $|\tilde{E}| \geq \frac{1}{2K}$, so that Proposition 6 can be deduced from

$$\int_{\tilde{E}} \int_{SSV(t)} |P_{1,t}(x)|^2 dx dt \leq c \frac{3^m}{K} \approx 3^{m/2}.$$

First, let us fix $t \in E$ and prove Proposition 5 using Salem's trick on

$$\int_0^{3^n} |P_1(x)|^2 dx :$$

Let $h(x) := (1 - |x|)\chi_{[-1,1]}(x)$, and note that $\hat{h}(\alpha) = C \frac{1 - \cos \alpha}{\alpha^2} > 0$. Then if we write $P_1 = 3^{m-n-1} \sum_{j=0}^{3^{n-m}} e^{i\alpha_j x}$, we get

$$\begin{aligned} \int_0^{3^n} |P_1(x)|^2 dx &\geq 2 \int_{-3^n}^{3^n} h(x) |P_1(x)|^2 dx \\ &\geq C(3^{m-n})^2 [3^n \cdot 3^{n-m} + \sum_{j \neq k; j, k=1}^{3^{n-m}} 3^n \hat{h}(3^n(\alpha_j - \alpha_k))] \geq C 3^m. \end{aligned}$$

To show that this is not concentrated on $[0, 3^{n-m}]$, we will use Theorem 2 and Lemma 27. We get

$$\begin{aligned} \int_0^{3^{n-m}} |P_1(x)|^2 dx &= \int_0^{3^{n-m}} |\widehat{\nu_{n-m}}(x)|^2 dx = 3^{2(m-n)} \int_0^{3^{n-m}} \left| \sum_{j=0}^{n-m} e^{i\alpha_j x} \right|^2 dx \\ &\leq CK \leq C 3^{\frac{m}{2}}. \end{aligned}$$

So now we have Proposition 5. The greater challenge will be Proposition 6.

2.3. Proposition 6: The estimate on $P_{1,t}^\sharp(x)$. Recall that $SSV(t) := \{x \in I : |P_{2,t}(x)| \leq 3^{-3m}\}$.

To get Proposition 5, we will split $P_{1,t}$ into two parts, $P_{1,t}^\sharp(x)$ and $P_{1,t}^\flat(x)$: a straightforward application of Lemma 27 to $P_{1,t}^\sharp(x)$ will get us part of the way there (for fixed t , the size of $SSV(t)$ does not overwhelm the average smallness of $P_{1,t}^\sharp(x)$), and the claims of Section 4 applied to $P_{1,t}^\flat(x)$ will further sharpen the final estimate to what we need.

Naturally, $P_{1,t}^b(x)$ and $P_{1,t}^\sharp(x)$ are defined as the medium and high frequency parts of $P_{1,t}(x)$. Below, $\ell := \alpha m$, for some large enough constant α :

$$P_{1,t}^b(x) := \prod_{k=n-m-\ell}^{n-m-1} \varphi_t(3^{-k}x) = \widehat{\nu_{\ell-1}}(3^{m+\ell-n}x),$$

$$P_{1,t}^\sharp(x) := \prod_{k=0}^{n-m-\ell-1} \varphi_t(3^{-k}x) = \hat{\nu}_{n-m-\ell-1}(x).$$

This is the claim of the subsection:

Proposition 7.

$$\int_{SSV(t)} |P_{1,t}^\sharp(x)|^2 dx \leq C'' K 3^m.$$

We will see in Section 5 that for each t , $SSV(t)$ is contained in $C \cdot 3^m$ neighborhoods of size $3^{n-m-\ell}$ around the complex zeroes λ_j of P_2 . (This is Corollary 23, which sounds plausible because the highest frequency among all factors of P_2 is about 3^{m-n} , and we are looking at an interval of length $< 3^n$. But much care has to be taken to show that the zeroes do not resonate between factors.)

Fix t . Let

$$I_j = [\lambda_j - 3^{n-m-\ell}, \lambda_j + 3^{n-m-\ell}], \quad (2.7)$$

$$\text{where } SSV(t) \subseteq \bigcup_j I_j \quad (2.8)$$

Choose j for which $\int_{I_j} |P_{1,t}^\sharp(x)|^2 dx$ is maximized. Then

$$\int_{SSV(t)} |P_{1,t}^\sharp(x)|^2 dx \leq C 3^m \int_{I_j} |P_{1,t}^\sharp(x)|^2 dx \leq C 3^m (3^{\ell+m-n})^2 \int_{I_j} \left| \sum_{k=0}^{n-m-\ell} e^{i\alpha_j x} \right|^2.$$

Recall $|I_j| \leq 2 \cdot 3^{n-m-\ell}$, so Lemma 27 and the definition of E give us Proposition 7.

2.4. Proposition 6: The estimate on $P_{1,t}^b(x)$. Of course we cannot just ignore $|P_{1,t}^b(x)|^2$ in $\int_{\tilde{E}} \int_{SSV(t)} |P_{1,t}^\sharp(x)|^2 |P_{1,t}^b(x)|^2 dx dt$, but one can bound it uniformly in each $I_j(t)$ by a Riesz product and then integrate in the t variable. Because the shape of \tilde{E} is rather complicated (see [8]), we will integrate our Riesz estimate on $|P_{1,t}^b(x)|$ for all $t \in [0, 1]$, where now x will be many functions $x_j(t)$ chosen to exhaust $SSV(t)$. When this is done, a factor of $K 3^{m/2}$ will be cancelled out in the right-hand side of Proposition 7, finally proving Proposition 6 and thus Theorem 3.

Now define

$$r(x) := \frac{7 + 2 \cos(x)}{9},$$

$$R(x) := \prod_{k=n-m-\ell}^{n-m-1} r(3^{-k} x)$$

The function $R(x)$ will estimate $|P_{1,t}^b(x_j(t))|$. The function R is $2\pi \cdot 3^{n-m-1}$ -periodic function. Note that its integral over a period is $(7/9)^\ell \cdot 3^{n-m-1}$. This is a general feature of Riesz products: when one integrates a full period, each factor can be identically replaced by its average. One can see this by changing the variable to get a 2π -periodic Riesz product and using lacunarity of the frequencies to compute the 0-th Fourier coefficient.

We will prove now

Lemma 8. $|\varphi_t(x)|^2 \leq \min(r(x), r(tx))$. In particular,

$$|P_{1,t}^b(x)|^2 \leq \min(R(x), R(tx)).$$

Proof. Notice that

$$|2 \cos \alpha + e^{i\beta}|^2 \leq 7 + 2 \cos(\alpha - \beta).$$

In fact,

$$|2 \cos \alpha + e^{i\beta}|^2 = 4 \cos^2 \alpha + 1 + 4 \cos \alpha \cos \beta \leq 5 + 2 \cos(\alpha + \beta) + 2 \cos(\alpha - \beta) \leq 7 + 2 \cos(\alpha - \beta).$$

One can write

$$9|\varphi_t(x)|^2 = |1 + e^{-itx} + e^{-ix}|^2 = |e^{-i\frac{t+1}{2}x} + 2 \cos(\frac{t-1}{2}x)|^2.$$

Having done this, one can replace β by $-\beta$ to get the other estimate. \square

We will see shortly that we need both Riesz estimates. Each has an associated change of variables, and the pair is sufficiently “separated away from simultaneous degeneracy.”

In Section 4, we will have occasion to consider $SSV(t)$ as a subset of $[3^{n-m}, 3^n] \times [-3^{n-m}, 3^{n-m}] \subset \mathbb{C}$. We will see that

$$SSV(t) \subseteq \bigcup_{j=1}^M B(\lambda_j(t), 3^{n-m-\ell}),$$

where the $\lambda_j(t)$ are the complex zeroes of $P_{2,t}$. They are in fact simple, depending differentiably on t , and no more than 3^{Bm} of them have some contact with the big

interval $I = [3^{n-m}, 3^n]$ (i.e., $M \leq 3^{Bm}$). Each $\lambda_j(t)$ has its t restricted to a time interval of size $c3^{-2m}$, which is called $D_r \cap \mathbb{R}$ (complex time $t \in D_r$ is considered in Section 4). So we divide $SSV(t) \cap I$ into the intersections of the neighborhoods of these zeroes with the real interval $[3^{n-m}, 3^n]$ to get the intervals $I_j(t)$. This consideration is made for each $r = 1, 2, \dots, R \leq C3^{2m}$ separately, since the time neighborhood D_r is of small enough size to control the number of zeroes $\lambda_j(t)$ entering and leaving the critical band $\mathbb{R} \times [-3^{-m}, 3^{-m}]$ during that time.

The $I_j(t)$ are centered at $x_j(t) = \text{Re}(\lambda_j(t))$, and have radius $3^{n-m-\ell}$. Lemma 19 says that within each $I_j(t)$ with t fixed, our Riesz estimates on $|P_{1,t}^b(x)|^2$ are absolutely comparable independent of x , and the constants of comparability depend on nothing. Further, we will define $R_j^*(t) := R(x_j(t))\chi_{U_j}(t) + R(tx_j(t))\chi_{V_j}(t)$, where U_j and V_j are open and cover $D_r \cap \mathbb{R}$, and $x_j(t)$, $tx_j(t)$ are differentiable on U_j and V_j respectively, with derivative bounded above and below by constant multiples of 3^{k-m} , where $k = n - m + 1, \dots, n$ denotes the factor of P_2 such that $\varphi_t(3^{-k}\lambda_j(t)) = 0$. U_j and V_j each have at most Cm components (See Section 4 for details).

Gathering all of this, consider a single j . Then

$$\begin{aligned} \int_{D_r \cap \mathbb{R}} |P_{1,t}^b(x_j(t))|^2 dt &\leq C \int_{t_0}^{t_0+3^{-2m}} R_j^*(t) dt \\ &\leq C3^{m-k} C' m \int_0^{3^{k-3m}} R(t) dt \leq C3^{m-k} C' m \int_0^{3^{n-m}} R(t) dt \\ &\leq C' m 3^{n-k} \left(\frac{7}{9}\right)^\ell \leq C' m 3^m \left(\frac{7}{9}\right)^\ell. \end{aligned}$$

This goes into the following, which uses Proposition 7:

$$\begin{aligned} \int_{D_r \cap \mathbb{R}} \int_{I_j(t)} |P_{1,t}^\sharp(x)|^2 |P_{1,t}^b(x)|^2 dx dt &\leq C \int_{D_r \cap \mathbb{R}} |P_{1,t}^b(x_j(t))|^2 \int_{I_j(t)} |P_{1,t}^\sharp(x)|^2 dx dt \\ &\leq C' m 3^m \left(\frac{7}{9}\right)^\ell \cdot C'' K 3^m. \end{aligned}$$

Summing over all j to cover $SSV(t)$ and then summing over all r to cover $[0, 1] \supseteq \tilde{E}$,

$$\int_{\tilde{E}} \int_{SSV(t)} |P_{1,t}(x)|^2 dx \leq 3^{Cm} C m K \left(\frac{7}{9}\right)^\ell \leq 3^{m/2}$$

The last inequality is true (and perhaps much better, of course) once one chooses α large enough and lets N (and therefore m and $\ell = \alpha m$) be large.

This completes the proof of Proposition 6 and of Theorem 3.

3. COMBINATORIAL PART

In this section, we show how Theorem 1 follows from Theorem 3.

First, let us define

$$\mathcal{L}_{\theta,N} := \text{proj}_{\theta} \mathcal{G}_N. \quad (3.1)$$

Theorem 9. *If $t \in E$ (see definition (2.4)), then $|\mathcal{L}_{\theta,NK^\beta}| \leq \frac{C}{K}$.*

Proof. Let us use θ instead of t and x for the space variable on the non-Fourier side, since we do not use Fourier analysis in this proof. Fix θ and let $F := A_K^* = \{x : f_N^*(x) \geq K\}$. We denote by N_x the line orthogonal to direction θ and passing through x . We can call it needle at x . For every $x \in F$ there are at least K triangles of size 3^{-r} , $r = r(x)$, $r \leq N$, intersecting N_x . Mark them. Run over all $x \in F$. Consider all marked triangles. Consider all 3^{-N} -triangles that are sub-triangles of marked ones. Call them “green”. Let U be a family of green triangles.

We want to show

$$\text{card } U \geq c \cdot K |F| 3^N, \quad (3.2)$$

$$|\text{proj}(\cup_{q \in U} q)| \leq \frac{C}{K} \text{card } U 3^{-N}, \quad (3.3)$$

Let $\phi := \sum_{q \in U} \chi_q$. Then

$$\int \phi dx = \text{card } U 3^{-N}.$$

Let M denote uncentered maximal function. To prove (3.3) it is enough to show that

$$q \in U \Rightarrow \text{proj } q \subset \{x : M\phi(x) > \frac{K}{C}\},$$

and then to use Hardy–Littlewood maximal theorem. But to prove this claim is easy. In fact, let $x \in \text{proj } q$, $q \in U$, then there exists Q —the maximal (by inclusion) marked triangle containing q . Consider $I := [x - 10 \ell(Q), x + 10 \ell(Q)]$. This segment contains the projections of at least K disjoint triangles $Q_1 := Q, Q_2, \dots, Q_K, \dots$, of the same sidelength, which intersect N_{x_0} , where x_0 is a point because of which $Q = Q_1$ was marked. (The reader should see that x_0 lies really well inside I .) So I contains the projections of at least $\frac{\ell(Q)}{\ell(q)} \cdot K$ green triangles. Whence,

$$\int_I \phi dx \geq \ell(q) \cdot \frac{\ell(Q)}{\ell(q)} \cdot K \geq \frac{1}{20} |I| K.$$

So

$$M\phi(x) > \frac{1}{20} K.$$

We proved (3.3).

Also we proved that $F \subset \{x : M\phi(x) \geq \frac{K}{20}\}$. Therefore, by Hardy–Littlewood maximal theorem

$$|F| \leq |\{x : M\phi(x) \geq \frac{K}{20}\}| \leq \frac{C \int \phi}{K} = C \text{card } U 3^{-N} K^{-1}.$$

This is (3.2).

Let us estimate $|\mathcal{L}_{\theta, N} K^\alpha|$ using (3.2) and (3.3). The first step:

$$\begin{aligned} |\mathcal{L}_{\theta, N}| &\leq |\text{proj}(\cup_{q \in U} q)| + 3^{-N}(3^N - \text{card } U) \leq \\ &\frac{C}{K} \text{card } U 3^{-N} + (3^N - \text{card } U) 3^{-N}. \end{aligned}$$

We do not touch the first term, but we improve the second term by using self-similar structure and going to step $2N$ (inside triangles which are not green there are “green” triangles of size 3^{-2N}). They are just self-similar copies of the original green triangles. Then we have the second step:

$$\begin{aligned} |\mathcal{L}_{\theta, N}| &\leq \frac{C}{K} \text{card } U 3^{-N} + \text{the rest} \leq \\ &\frac{C}{K} \text{card } U 3^{-N} + (3^N - \text{card } U) \frac{C}{K} \text{card } U 3^{-2N} + (3^N - \text{card } U)^2 3^{-2N}. \end{aligned}$$

Now we leave first two terms alone and having $(3^N - \text{card } U)^2$ triangles of size 3^{-2N} we find again “green” triangles inside each of those, now green triangles of size 3^{-3N} . They are just self-similar copies of original green triangles.

Then we have the third step:

$$\begin{aligned} |\mathcal{L}_{\theta, 3N}| &\leq \frac{C}{K} \text{card } U 3^{-N} + (3^N - \text{card } U) \frac{C}{K} \text{card } U 3^{-2N} + \text{the rest} \leq \\ &\frac{C}{K} \text{card } U 3^{-N} + (3^N - \text{card } U) \frac{C}{K} \text{card } U 3^{-2N} + (3^N - \text{card } U)^2 \frac{C}{K} \text{card } U 3^{-2N} + \\ &(3^N - \text{card } U)^3 3^{-3N}. \end{aligned}$$

After the l -th step:

$$\begin{aligned} |\mathcal{L}_{\theta, lN}| &\leq \frac{C}{K} \text{card } U 3^{-N} (1 + (3^N - \text{card } U) 3^{-N} + \dots \\ &+ (3^N - \text{card } U)^{l-1} 3^{-(l-1)N}) + (3^N - \text{card } U)^l 3^{-(lN)}. \end{aligned}$$

So

$$|\mathcal{L}_{\theta, lN}| \leq \frac{C}{K} \text{card } U 3^{-N} \frac{(1 - (1 - \frac{\text{card } U}{3^N})^l)}{(1 - (1 - \frac{\text{card } U}{3^N}))} + e^{-\frac{\text{card } U}{3^N} l} =: I + II.$$

Notice that by (3.2) $II \leq e^{-K|F|l} \leq e^{-K}$ if the step l is chosen to be $l = 1/|F| \leq K^\beta$. However, we always have $I \leq \frac{C}{K}$. So Theorem 9 is completely proved. \square

From Theorems 3 and 9, it is not hard to get Theorem 1.

4. THE COMPLEX ANALYTIC PART

4.1. Elementary facts about φ . In this section, we investigate the various nice properties of φ , considered as a function of the complex variable $z = x + iy$, with $x > 0$. We will work mostly with

$$\tilde{\varphi}_t(z) := 3\varphi_t(z) = 1 + e^{-itz} + e^{-iz}.$$

Recall that $t \in [0, 1]$ earlier. We also complexify t : $t = u + iv \in T$, where $T := [0 - 3^{-m}, 1 + 3^{-m}] \times [-3^{-m}, 3^{-m}]$. Define also

$$\tilde{T} := T \setminus ([1/2 - 3^{-m}, 1/2 + 3^{-m}] \times [-3^{-m}, 3^{-m}]). \quad (4.1)$$

This deletion is motivated by Section 4.2.

Note that we are trying to control the zeroes of

$$P_{2,t}(z) = \prod_{k=n-m+1}^n \varphi_t(3^{-k}z) = \widehat{\nu}_{m-1}(3^{m-1-n}z),$$

for $\text{Re}(z) \in [3^{n-m}, 3^n]$. For this purpose, it suffices to consider $\tilde{\varphi}_t$ restricted to not far from

$$z \in \tilde{I} := [3^{-m}, 3^m]. \quad (4.2)$$

Notice that at the end, we will have to multiply by 3^k to get the location of the zeroes back to where they belong in the big picture of Section 2. Call the zeroes of $\tilde{\varphi}_t$ $\tilde{\lambda}$, and only call the zeroes λ when they are regarded as zeroes of the factors of $\phi(3^{-k}\cdot)$.

To use Blaschke estimates along the real line, we need for fixed t that x is never far from z such that $|\tilde{\varphi}_t(z)| > 1/2$.

Lemma 10. *There exists $H > 0$ such that $\forall t \in T, x > 0$*

$$\max_{|z-x| \leq H} |\tilde{\varphi}_t(z)| \geq 1/2.$$

Proof. First, consider t real. Notice that for $t \leq 1/2$, e^{-iz} is the dominant summand for $y \geq H$, and for $t \geq 1/2$, 1 is dominant for $y \leq -H$. Thus we are never more than the distance H from a point z at which $|\tilde{\varphi}_t| > 1/2$.

For t complex, we can write $\tilde{\varphi}_t(z) = 1 + c_1 e^{vx+uy} + c_2 e^y$, for some $|c_1| = |c_2| = 1$ depending on z, t . Since $x \in \tilde{I}$ and $t \in \tilde{T}$, $xv \leq C$. By choosing a larger value of H if needed, we can make either 1 or $c_2 e^y$ dominant like before. \square

Lemma 11. *There exists an absolute constant M such that in $B(x, 1)$, $\tilde{\varphi}_t$ has at most M complex zeroes $\tilde{\lambda}_j$. Further, the set of z where $|\tilde{\varphi}_t(z)| \leq \varepsilon \ll 1$ is contained in*

$$\bigcup_j B(\tilde{\lambda}_j, C\varepsilon^M).$$

Proof. To use Lemma 29, we need the previous lemma, and we need to bound $|\tilde{\varphi}_t(z)|$ from above on a neighborhood of the point z_0 where $|\tilde{\varphi}_t(z_0)| > 1/2$. This neighborhood needs to contain $B(x, 1)$. This is not hard, either: $vx \leq C$, and $|y| \leq H$ at the the point z_0 , so $|\tilde{\varphi}_t(z)|$ is easily bounded absolutely in a $2H$ -neighborhood of z_0 . The other claim is Lemma 30. \square

4.2. Branch points of $\tilde{\varphi}_t$ and analytic continuation of zeroes. For $t \in T$ and $z \in \mathbb{R} \times [-H, H]$, we call the pair (t, z) a *branch point* of $\tilde{\varphi}_t$ if

$$\begin{cases} \tilde{\varphi}_t(z) = 0 \\ \frac{\partial}{\partial z} \tilde{\varphi}_t(z) = 0. \end{cases} \quad (4.3)$$

Lemma 12. *There are no branch points such that t is a real number in $[0, 1]$.*

Proof. If (4.3) is valid then

$$\begin{cases} e^{-itz} = -1 - e^{-iz} \\ te^{-itz} + e^{-iz} = 0. \end{cases} \quad (4.4)$$

Hence $e^{-iz}(1-t) - t = 0$. Of course $t = 0, 1$ are impossible. So $e^{-iz} = \frac{t}{1-t}$. Doing the other substitution, one gets $e^{-itz} = \frac{1}{t-1}$.

Taking absolute values:

$$\begin{cases} e^y = \frac{t}{1-t} \\ e^{ty} = \frac{1}{1-t} \end{cases} \quad (4.5)$$

$0 < t < 1$, so $e^y < e^{ty} \Rightarrow y < 0$.

But then $\frac{1}{1-t} = e^{ty} < 1$, a contradiction. \square

This allows us to analytically continue zeroes: $\tilde{\lambda}_j(t)$, holomorphic in t on some neighborhood T_j of $[0, 1]$, satisfying $\tilde{\varphi}_t(\tilde{\lambda}_j) = 0$. But we would like to control $|\lambda'(t)|$, so we restrict to \tilde{T} ⁽⁴⁾, where estimates are easier to come by. Note that if z is a zero of $\tilde{\varphi}_t$, $\text{Im}(z) \leq 2 \cdot 3^{-m}$, and $t \in \tilde{T} \cap \mathbb{R}$, we have

$$3 \geq |\tilde{\varphi}'_t(z)| = |(1-t)e^{itz} - t| \geq c|1-2t| \geq c3^{-m}. \quad (4.6)$$

Lemma 13. *For this lemma, the subscripts z and τ will denote partial derivatives. Let $\Phi(z, \tau)$ be holomorphic in both complex variables. Let $\Phi(z_0, \tau_0) = 0$ and $\Phi_z(z_0, \tau_0) \neq 0$. Suppose that on some neighborhood $B(z_0, \delta_1) \times B(\tau_0, \delta_2)$, one has*

$$|\Phi_z(z, \tau)| \geq \eta \text{ and } |\Phi_\tau(z, \tau)| \leq C.$$

Then there exists a unique holomorphic function $\tilde{\lambda} : B(w_0, \varepsilon) \rightarrow \mathbb{C}$ such that

$$\begin{cases} \tilde{\lambda}(\tau_0) = z_0 \\ \Phi(\tilde{\lambda}(w), w) = 0. \end{cases}$$

$$\text{Also, } \tilde{\lambda}'(\tau) = -\frac{\Phi_\tau(\tilde{\lambda}(\tau), \tau)}{\Phi_z(\tilde{\lambda}(\tau), \tau)} \text{ and } \varepsilon = \min\{c\delta_1\eta, \delta_2\}.$$

Applied to our case, Lemma 13 and 4.6 gives us this estimate for $\tilde{\lambda}'_j(t)$, when $|\text{Im}(\tilde{\lambda}_j(t))| \leq 2 \cdot 3^{-m}$:

$$|\tilde{\lambda}'_j(t)| \leq C3^m \quad (4.7)$$

This is still somewhat fast, but we will see later in Section 5 that we only need be concerned when $|\text{Im}(\tilde{\lambda}_j(t))| \leq 3^{-m}$. In this case, we say $\lambda_j(t)$ is in the critical band \tilde{Q} of radius 3^{-m} around \tilde{I} ⁽⁵⁾. Consider also the band \tilde{R} of radius $2 \cdot 3^{-m}$ around \tilde{I} . The factor 2 establishes a buffer through which it takes at least $c3^{-2m}$ “seconds” to pass before entering the critical band from the outside. So if we count zeros in \tilde{R} at an initial real time t_0 , only those zeroes can enter the critical band \tilde{Q} during this small interval of complex time. This is the content of the next lemma.

Let us state this as a lemma. Cover $\tilde{T} \cap \mathbb{R}$ by discs

$$D_r := B(t_r, c3^{-2m}), \quad t_r := cr3^{-2m}, \quad r = 1, 2, \dots, R. \quad (4.8)$$

Lemma 14. *Fix an r . Let $(z_0, t_0) \in \tilde{Q} \times (D_r \cap \mathbb{R})$ be a pair such that $\tilde{\varphi}_{t_0}(z_0) = 0$. Then all such pairs belong to a union of paths $\tilde{\lambda}_j(t)$, for $j = 1, 2, \dots, J$, where $J \leq C3^m$.*

⁴Definition (4.1)

⁵Definition (4.2)

Proof. For any such pair (z_0, t_0) , one can analytically continue z_0 as a function $\tilde{\lambda}(t)$ in a disc of radius $c3^{-2m}$ around t_0 (this is Lemma 13). Such a disc meets t_r . So $z_0 = \tilde{\lambda}(t_0)$, for some $\tilde{\lambda} : D_r \rightarrow \mathbb{C}$.

$\tilde{\lambda}_j(t_r) \in \tilde{R}$ because of (4.7) and because $\tilde{\lambda}_j(t_0) \in \tilde{Q}$. Because of Lemma 11, at the initial time t_r there were at most $C3^m$ such $\tilde{\lambda}_j$ within distance 1 of $\tilde{I} \subseteq \tilde{R}$. So at time t_r , we can number the zeroes $\tilde{\lambda}_j(t) \in \tilde{R}$, $j = 1, 2, \dots, J$, and for each $t \in \mathcal{D}_r$, all zeroes of $\tilde{\varphi}_t$ in \tilde{Q} lie along one of these paths. \square

4.3. Holomorphic extension of the real parts of the $\tilde{\lambda}_j$. Let

$$\tilde{x}_j(t) := \frac{1}{2}(\tilde{\lambda}_j(t) + \bar{\tilde{\lambda}}_j(\bar{t})).$$

Then the \tilde{x}_j are holomorphic, and for t real, $\tilde{x}_j(t) = \text{Re}(\tilde{\lambda}_j(t))$. We use $\tilde{x}(t)$ here, analogous to the remark about $\tilde{\lambda}(t)$. We will remove the tilde when we change variables by 3^k and adapt the zeroes back to the factors of P_2 .

In Section 2, we consider functions

$$g_{1,j}(t) = \tilde{x}_j(t) \text{ and } g_{2,j}(t) = t\tilde{x}_j(t). \quad (4.9)$$

These are the changes of variable in the Riesz estimates. For fixed j , we have (4.7), so

$$|g_{i,j}|, |g'_{i,j}| \leq C3^m. \quad (4.10)$$

We sometimes drop the j when it is considered fixed in a context.

Lemma 15. *Fix i and j . Within D_r on which g_i is defined, one of the following is true:*

$$\|g'_i\|_\infty \leq C3^{-m} \quad (4.11)$$

$$\#\{t : g'_i = 0\} \leq Cm. \quad (4.12)$$

Proof. Suppose 4.11 is false. Divide g'_i by $C3^{-m}$, so that Lemma 29 applies to this new function. Then the conclusion is exactly (4.12). \square

This is good because of the following:

Lemma 16. *For fixed j , it is impossible for $|g'_i(t)| \leq C3^{-m}$ to happen for $i = 1, 2$ simultaneously at t . In particular, (4.11) cannot happen for $i = 1, 2$ simultaneously.*

Proof. Fix j and fix $t \in \tilde{T}$. Suppose $|g'_1(t)|, |g'_2(t)| < \varepsilon = C3^{-m}$. By direct computation from the definition (4.9), one gets $|\tilde{x}_j(t)| = |g'_1(t) - tg'_2(t)| \leq 2\varepsilon$. But $|\tilde{x}_j(t)| \geq c3^{-m}$. It follows that $\varepsilon \geq c'3^{-2m}$. \square

Corollary 17. *Fix j , and let $\tilde{\lambda}_j$ be defined on D_r . For at least one of $i = 1, 2$, (4.12) holds. At each such zero t of g_i , $|g'_{3-i}(t)| \geq c3^{-m}$.*

We need a little more. First, notice that for real t , the $g_{i,j}(t)$ are real, as are the $g'_{i,j}(t)$.

Lemma 18. *For each j , each $D_r \cap \mathbb{R}$ can be covered by real open sets \tilde{U}_j, \tilde{V}_j so that $|g'_{1,j}(t)| \geq c3^{-m}$ on \tilde{U}_j and $|g'_{2,j}(t)| \geq c3^{-m}$ on \tilde{V}_j . Further, \tilde{U}_j and \tilde{V}_j are unions of at most Cm open intervals, where C does not depend on anything.*

Proof. In all of D_r , consider $p = g_1 + g_2$ and $m = g_1 - g_2$, and imitate the last two lemmas.

$|p' + m'| = |g'_1|$ and $|p' - m'| = 2|g'_2|$, so it is impossible for $|p'|, |q'| \leq c3^{-m}$ simultaneously. Now we have cases:

$$\begin{cases} \|p'\|_\infty \leq c3^{-m} \\ \|m'\|_\infty \leq c3^{-m} \\ \#\{t \in D_r : p' = 0 \text{ or } m' = 0\} \leq Cm \end{cases}$$

The third case exhausts the remaining possibilities exactly as in Lemma 4.12. Note that in either of the first two cases, $\|g'_1| - |g'_2|\| \leq \frac{c}{2}3^{-m}$, so by Corollary 17, $|g'_1|, |g'_2| \geq \frac{c}{2}3^{-m}$ throughout D_r . So by perhaps changing the constant c , these cases are settled. (\tilde{U} and/or \tilde{V} may be taken to be $D_r \cap \mathbb{R}$.)

In the last case, we now restrict the above complex analytic information to the real line, and remember that the g'_i are real. In particular, there are only Cm such $t \in D_r \cap \mathbb{R}$. Away from such t , we are in an interval where either $|g'_1| > |g'_2|$ or the opposite. But by Lemma 17, the larger of the two is always larger than $c3^{-m}$, and so the interval is a component of \tilde{U} or \tilde{V} , accordingly. \square

4.4. Rescaling back, and uniform Riesz bounds. Let $\lambda_j(t) = 3^k \tilde{\lambda}_j(t)$, $x_j(t) = 3^k \tilde{x}_j(t)$, etc. Now everything moves 3^k times as fast and has neighborhoods 3^k times as large, and possibly shows up in the interval $I = [3^{n-m}, 3^n]$ once for each $k = n - m + 1, \dots, n$. Tildes can be removed from everything in this way, and a copy

gets plugged into Section 2 for each such k . So now we will know how to integrate the function $R_j^*(t) = R_j(x_j(t))\chi_{U_j}(t) + R_j(tx_j(t))\chi_{V_j}(t)$:

Lemma 19. *For all $t \in D_r \cap \mathbb{R}$, for each of its j , and for each $x \in I_j(t)$, one has*

$$R_j^*(t) \leq C[R_j(x)\chi_{U_j}(t) + R_j(tx)\chi_{V_j}(t)]$$

Proof. Recall:

$$R(x) := \prod_{k=n-m-\ell}^{n-m-1} r(3^{-k}x), \quad r(x) := \frac{7 + 2\cos(x)}{9},$$

and $I_j(t) := [x_j(t) - 3^{n-m-\ell}, x_j(t) + 3^{n-m-\ell}]$.

The χ_{U_j} , χ_{V_j} truncate the small values out of our considerations. In each Riesz product, each factor belongs to $[\frac{5}{9}, 1]$, and in fact one could let $k \rightarrow +\infty$ in the above product and get geometric convergence, uniform in x on the given interval. So one gets a constant like $(\frac{9}{5})^C$. \square

The next section explains why considering the zeroes of the different factors of P_2 separately does no real harm to the main argument.

5. ANALYTIC TILING

5.1. Preamble. In Section 4, only a single factor of $\tilde{\varphi}_t$ was considered at a time. In this part, we show that the product

$$\Phi_m(z) = \prod_{k=1}^m \tilde{\varphi}_t(3^{-k}z)$$

has a well-behaved **set of small values**: only one factor may be critically small at a given time and place, and the **product** of the remaining terms is no smaller than 3^{-m} , so one can estimate integrals along $SSV(t)$ by considering the zeroes of each factor separately.

Something much worse could have happened: all or most terms could have been less than 3^{-m} simultaneously, so that outside of the neighborhood, one would only have the estimate $\Phi_m(z) \geq (3^{-m})^m = 3^{-m^2}$. Our set of small values would have been a set of very small values indeed, resulting in the weaker final estimate $Fav(\mathcal{G}_n) \leq e^{-c\sqrt{\log n}}$. See also the discussion, Section 8.

5.2. Result. We will regard $t \in [0, 1]$ as fixed here, reclaiming the subscript for other purposes. First, some definitions. $c > 1$ will be an absolute constant. δ will be a small enough absolute constant, $x_0 \in \mathbb{R}$, $m \in \mathbb{N}$ is large. $R := [x_0 - \delta, x_0 + \delta] \times [-\delta, \delta]$, and $2R := [x_0 - 2\delta, x_0 + 2\delta] \times [-2\delta, 2\delta]$

$$\tilde{\varphi}(z) := 1 + e^{-iz} + e^{-itz}$$

$$\tilde{\varphi}_k(z) := \tilde{\varphi}(3^{-k}z)$$

$$\Phi(z) := \prod_{k=0}^m \tilde{\varphi}_k(z)$$

Also of interest will be another function, $\Phi_{k_0} := \Phi/\tilde{\varphi}_{k_0}$. The most important thing to prove, and the place where so-called analytic tiling comes into play, is in the proof of

Proposition 20. $\forall z$ with $|Im(z)| \leq \delta$, $\max_{k_0} |\Phi_{k_0}(z)| \geq \delta^m$. Further, say that k_0 is **critical** if $\exists z_0 \in R$ with $|\tilde{\varphi}_{k_0}(z_0)| < \delta^m$. Then a critical k_0 is unique whenever it exists.

This will lead to the following:

Proposition 21. Let M, c be sufficiently large absolute constants. Let

$$SSV(R) := \{z \in R : |\Phi(z)| < \delta^{2m}\}$$

If there is a critical k_0 , let

$$SSV_{k_0}(R) := \{z \in R : |\tilde{\varphi}_{k_0}(z)| < \delta^m\}.$$

Then $SSV(R) = \emptyset$ if there is no critical k_0 , and otherwise

$$SSV(R) \subseteq SSV_{k_0}(R) \subseteq \bigcup_j B(\lambda_j, c(3\delta)^{m/M}),$$

where the λ_j are the zeroes of $\tilde{\varphi}_{k_0}$ in $2R$ ⁶.

Proof. If there is no critical k_0 , then for each $z \in R$ there is some k_0 such that $|\Phi(z)| = |\Phi_{k_0}(z)\tilde{\varphi}_{k_0}(z)| \geq \delta^{2m}$. Thus $SSV(R) = \emptyset$. If there is a critical k_0 , then $|\Phi| = |\Phi_{k_0}\tilde{\varphi}_{k_0}| \geq \delta^m|\tilde{\varphi}_{k_0}|$ shows that $SSV(R) \subseteq SSV_{k_0}(R)$. By the Blaschke estimate (Lemma 11), $\tilde{\varphi}_{k_0}$ has at most M zeroes λ_j in $B(x_0, 3^k) \subseteq B(x_0, 3^m)$, and

$$SSV_{k_0}(R) \subseteq \bigcup_j B(\lambda_j, C(3\delta)^{m/M}).$$

⁶Here $2R$ is concentric with R

□

Corollary 22. *M did not depend on δ , so we can pick δ to be an absolute constant such that $c(3\delta)^{m/M} < 3^{-m-\alpha m} = 3^{-m-\ell}$. Thus in the horizontal strip of thickness 2δ , the neighborhoods of small values have size $< 3^{-m-\ell}$, and there are no more than $2M/\delta$ of them per unit interval.*

Corollary 23. *In the setting of Section 2, this says that $SSV(t)$ is contained in $C3^m$ intervals of size $3^{n-m-\ell}$. This is by changing variables and by going back to φ instead of $\tilde{\varphi}$ by multiplying 3^{n-m} back in.*

The main idea behind Proposition 20 is to analyze the stability under perturbations of the solution to the following equations, unique up to swapping w_1 with w_2 :

$$\begin{cases} |w_1| = |w_2| = 1 \\ 1 + w_1 + w_2 = 0 \end{cases}$$

Clearly $w_j = e^{2\pi i j/3}$. What is interesting about this is that $1 + (w_1)^{3^k} + (w_2)^{3^k} = 3 \forall k = 1, 2, 3, \dots$. This is stable, if use k to control the size of the perturbations of the w_j .

Lemma 24. *Let $|y_1|, |y_2| \leq c\delta 3^{-k'}$, and suppose that w_1, w_2 satisfy:*

$$\begin{cases} |w_j| = e^{y_j} \\ 1 + w_1 + w_2 = 0 \end{cases}$$

*Then for $1 \leq k \leq k' + 1$, $|1 + (w_1)^{3^k} + (w_2)^{3^k}| \geq 2$.
(In fact, $\text{Re}(1 + (w_1)^{3^k} + (w_2)^{3^k}) \geq 2$)*

Proof. Write $w_j = e^{ix_j + iy_j}$. Without loss of generality, $|x_j - 2\pi j/3| \leq C\delta 3^{-k'}$. Thus $(w_j)^{3^k}$ have the appropriate arguments and magnitudes. □

Corollary 25. *If $|\tilde{\varphi}_{k'}(z)| < c\delta 3^{-k^*}$, then $\forall k = k' - 1, \dots, k' - k^* - 1$, $|\tilde{\varphi}_k(z)| \geq 2$*

Proof. Let $w_1 = e^{i3^{-k'}z}$, $w_2 = e^{it3^{-k'}z}$. □

Finally, let us prove Proposition 20.

By induction. $m = 0$ is clear. $\delta \ll 1$; in particular, $\delta < \frac{1}{3}$. Assume Proposition 20 for $m - 1$. Fix z in R . If $|\tilde{\varphi}_m(z)| \leq \delta^m$, then $k_0 = m$, since all other factors must be at least 2 due to Corollary 25. Now let $\delta^{j+1} \leq |\tilde{\varphi}_m(z)| \leq \delta^j$, for some $j < m$

(or just induct if $|\tilde{\varphi}_m(z)| > 1$). Then $|\tilde{\varphi}_{m-k}(z)| \geq 2$ for all $k = 1, \dots, j$, again by Corollary 25. Thus

$$\left| \prod_{k=m-j}^m \tilde{\varphi}_k(z) \right| \geq \delta^{j+1}.$$

By the induction hypothesis,

$$\exists k_0 : \left| \prod_{k=0, k \neq k_0}^{m-j-1} \tilde{\varphi}_k(z) \right| \geq \delta^{m-j-1}$$

These two inequalities yield $|\Phi_{k_0}| \geq \delta^m$.

Next, we show that there can be at most one critical k_0 . If there is a critical k_0 , consider the largest. This means that $\exists z_0 \in SSV_{k_0}(R)$. So z_0 lies in a small neighborhood of a zero λ of $\tilde{\varphi}_{k_0}$ in $2R$ (concentric). Since $|\tilde{\varphi}'_{k_0}| \leq 2 \cdot 3^{-k_0}$, it follows that $|\tilde{\varphi}_{k_0}| \leq c\delta 3^{-k_0}$ in R . Thus $|\tilde{\varphi}_k(z)| \geq 2$ for all $k < k_0$ and for all $z \in R$. k_0 was chosen to be the largest, so it is unique. \square

6. SOME IMPORTANT STANDARD LEMMAS

There are a few important lemmas which we have appealed to repeatedly. The first claim, Lemma 26, uses the Carleson imbedding theorem. A stronger version, Lemma 27, uses general H^2 theory. Its importance lies in its ability to establish a key relationship between the level sets of $f_{n,t}$ and the L^2 norm of $\widehat{f_{n,t}}$. This is because the Fourier transform changes the centers of intervals into the frequencies of an exponential polynomial.

The second claim we split into Lemmas 29 and 30. Given a bounded holomorphic function on the disc, its supremum, and an interior non-zero value, these lemmas bound the number of zeroes and contain the set of small values within certain neighborhoods of these zeroes.

6.1. In the spirit of the Carleson imbedding theorem.

Lemma 26. *Let $j = 1, 2, \dots, k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$. Then*

$$\int_0^1 \left| \sum_{j=1}^k c_j e^{i\alpha_j y} \right|^2 dy \leq C k \cdot \sup_{I \text{ a unit interval}} \#\{A \cap I\}.$$

Proof. Let $A_1 := \{\mu = \alpha + i : \alpha \in A\}$. Let $\nu := \sum_{\mu \in A_1} \delta_\mu$. This is a measure in \mathbb{C}_+ . Obviously its Carleson constant

$$\|\nu\|_C := \sup_{J \subset \mathbb{R}, J \text{ is an interval}} \frac{\nu(J \times [0, |J|])}{|J|}$$

can be estimated as follows

$$\|\nu\|_C \leq 2 \sup_{I \text{ a unit interval}} \#\{A \cap I\}. \quad (6.1)$$

Recall that

$$\forall f \in H^2(\mathbb{C}_+) \int_{\mathbb{C}_+} |f(z)|^2 d\nu(z) \leq C_0 \|\nu\|_C \|f\|_{H^2}^2, \quad (6.2)$$

where C_0 is an absolute constant. Now we compute

$$\begin{aligned} \int_0^1 \left| \sum_{j=1}^k c_j e^{i\alpha_j y} \right|^2 dy &\leq e^2 \int_0^1 \left| \sum_{j=1}^k c_j e^{i(\alpha_j + i)y} \right|^2 dy \leq \\ &e^2 \int_0^\infty \left| \sum_{j=1}^k c_j e^{i(\alpha_j + i)y} \right|^2 dy = e^2 \int_{\mathbb{R}} \left| \sum_{\mu \in A_1} \frac{c_\mu}{x - \mu} \right|^2, \end{aligned}$$

where $c_\mu := c_j$ for $\mu = \alpha_j + i$. The last equality is by Plancherel's theorem.

We continue

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{\mu \in A_1} \frac{c_\mu}{x - \mu} \right|^2 &= \sup_{f \in H^2(\mathbb{C}_+), \|f\|_2 \leq 1} \left| \left\langle f, \sum_{\mu \in A_1} \frac{c_\mu}{x - \mu} \right\rangle \right|^2 = \\ &4\pi^2 \sup_{f \in H^2(\mathbb{C}_+), \|f\|_2 \leq 1} \left| \sum_{\mu \in A_1} c_\mu f(\mu) \right|^2 \leq C \#\{A_1\} \sup_{f \in H^2(\mathbb{C}_+), \|f\|_2 \leq 1} \sum_{\mu \in A_1} |f(\mu)|^2 \leq \\ &C \#\{A\} \sup_{f \in H^2(\mathbb{C}_+), \|f\|_2 \leq 1} \int_{\mathbb{C}_+} |f(z)|^2 d\nu(z) \leq 2C_0 C \#\{A\} \sup_{I \text{ a unit interval}} \#\{A \cap I\}. \end{aligned}$$

This is by (6.7) and (6.1). The lemma is proved. \square

Now we are going to prove a stonger assertion by a simpler approach. This stronger assertion is what is used in the main part of the article.

Lemma 27. *Let $j = 1, 2, \dots, k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$. Then Suppose*

$$\int_{\mathbb{R}} \left(\sum_{\alpha \in A} \chi_{[\alpha-1, \alpha+1]}(x) \right)^2 dx \leq S, \quad (6.3)$$

Then there exists an absolute constant C

$$\int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} \right|^2 dy \leq C S. \quad (6.4)$$

Of course, one can change variables and get:

Corollary 28. *Let $j = 1, 2, \dots, k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$, and let $\delta > 0$. Suppose*

$$\int_{\mathbb{R}} \left(\sum_{\alpha \in A} \chi_{[\alpha-\delta, \alpha+\delta]}(x) \right)^2 dx \leq S, \quad (6.5)$$

Then there exists an absolute constant C

$$\int_a^{a+\delta^{-1}} \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} \right|^2 dy \leq C S / \delta^2. \quad (6.6)$$

Remark. Lemma 27 is obviously stronger than Lemma 26. In fact, let S_0 be the maximal number of points A in any unit interval. Then

$$f(x) := \sum_{\alpha \in A} \chi_{[\alpha-1, \alpha+1]}(x) \leq 2S_0.$$

Now $\int_{\mathbb{R}} f^2(x) dx \leq 4kS_0$, where k as above is the cardinality of A . We can put now $S := 4kS_0$, apply Lemma 27 and get the conclusion of Lemma 26. The proof of Lemma 27 does not require the Carleson imbedding theorem. Here it is.

Proof. Using Plancherel's theorem we write

$$\begin{aligned} \int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} \right|^2 dy &\leq e \int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i(\alpha+i)y} \right|^2 dy \leq e \int_0^\infty \left| \sum_{\alpha \in A} c_\alpha e^{i(\alpha+i)y} \right|^2 dy = \\ &e \int_{\mathbb{R}} \left| \sum_{\alpha \in A} \frac{c_\alpha}{\alpha + i - x} \right|^2 dx. \end{aligned}$$

Recall that

$$H^2(\mathbb{C}_+) \text{ is orthogonal to } \overline{H^2(\mathbb{C}_+)} \quad (6.7)$$

Now we continue

$$\begin{aligned} &\int_{\mathbb{R}} \left| \sum_{\alpha \in A} \frac{c_\alpha}{\alpha + i - x} \right|^2 dx \leq \\ &\int_{\mathbb{R}} \left| \sum_{\alpha \in A} \frac{c_\alpha}{\alpha + i - x} - \sum_{\alpha \in A} \frac{c_\alpha}{\alpha - i - x} \right|^2 dx = \\ &\frac{\pi}{2} \int_{\mathbb{R}} \left| \sum_{\alpha \in A} c_\alpha P_1(\alpha - x) \right|^2 dx, \end{aligned}$$

where P_1 is the Poisson kernel in the half-plane C_+ at height $h = 1$:

$$P_h(x) := \frac{1}{\pi} \frac{h}{h^2 + x^2}.$$

We continue by noticing that $P_1 * \chi_{[\lambda-1, \lambda+1]}(x) \geq c P_1(\lambda - x)$ with absolute positive c . This is an elementary calculation, or, if one wishes, the Harnack's inequality. Now we can continue

$$\int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} dy \right|^2 \leq \frac{\pi e}{2c} \int_{\mathbb{R}} \left| (P_1 * \sum_{\alpha \in A} c_\alpha \chi_{[\alpha-1, \alpha+1]})(x) \right| dx.$$

Now we use the fact that $f \rightarrow P_1 * f$ is a contraction in $L^2(\mathbb{R})$. So

$$\int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} dy \right|^2 \leq \frac{\pi e}{2c} \int_{\mathbb{R}} \left| \sum_{\alpha \in A} c_\alpha \chi_{[\alpha-1, \alpha+1]}(x) \right| dx \leq C S.$$

The lemma is proved. □

6.2. A Blaschke estimate.

Lemma 29. *Let D be the closed unit disc in \mathbb{C} . Suppose ϕ is holomorphic in an open neighborhood of D , $|\phi(0)| \geq 1$, and the zeroes of ϕ in $\frac{1}{2}D$ are given by $\lambda_1, \lambda_2, \dots, \lambda_M$. Let $C = \|\phi\|_{L^\infty(D)}$. Then $M \leq \log_2(C)$.*

Proof. Let

$$B(z) = \prod_{k=1}^M \frac{z - \lambda_k}{1 - \bar{\lambda}_k z}.$$

Then $|B| \leq 1$ on D , with $=$ on the boundary. If we let $g := \frac{\phi}{B}$, then g is holomorphic and nonzero on $\frac{1}{2}D$, and $|g(e^{i\theta})| \leq C \forall \theta \in [0, 2\pi]$. Thus $|g(0)| \leq C$ by the maximum modulus principle. So we have

$$C \geq |g(0)| = \frac{|\phi(0)|}{|B(0)|} \geq \prod_{k=1}^M \frac{1}{|\lambda_k|} \geq 2^M.$$

□

Lemma 30. *In the same setting as Theorem 29, the following is also true for all $\delta \in (0, 1/3)$: $\{z \in \frac{1}{4}D : |\phi| < \delta\} \subseteq \bigcup_{1 \leq k \leq M} B(\lambda_k, \varepsilon)$, where*

$$\varepsilon := \frac{9}{16} (3\delta)^{1/M} \leq \frac{9}{16} (3\delta)^{1/\log_2(C)}.$$

Proof. Let $\delta \in (0, 1/3)$, and let $z \in \frac{1}{4}D$ such that $|z - \lambda_k| > \varepsilon \forall k$. Note that g is harmonic and nonzero on $\frac{1}{2}D$ with $|g(0)| \geq 2^M$. Thus Harnack's inequality ensures that $|g| \geq \frac{1}{3}2^M$ on $\frac{1}{4}D$, so there

$$|\phi(z)| \geq |g(z)B(z)| \geq \frac{1}{3}2^M \prod_{k=1}^M \left| \frac{z - \lambda_k}{1 - \lambda_k z} \right| \geq \left(\frac{16\varepsilon}{9} \right)^M \frac{1}{3} = \delta.$$

We can conclude the proof by the contrapositive. \square

7. COMBINATORIAL THEOREM

Theorem 31. *For this section, regard the set E from Section 2 as parameterized by θ , and use the variable x instead of s on the non-Fourier side, since we will not work on the Fourier side at all during this section.*

Let $\theta \in E$. Then

$$\max_{n: 0 \leq n \leq N} \|f_{n,\theta}\|_{L^2(\mathbb{R})}^2 \leq CK.$$

To prove this we first need the following claim, which is the main combinatorial assertion of this article. It repeats the one in [14] but we give a slightly different proof.

We fix a direction θ , we think that the line ℓ_{θ} on which we project is \mathbb{R} . If $x \in \mathbb{R}$ then by N_x we denote the line orthogonal to \mathbb{R} and passing through point x , we call N_x a needle. By F_L we denote $\{x \in \mathbb{R} : f_N^*(x) := \max_{0 \leq n \leq N} f_{n,\theta}(x) > L\}$.

Theorem 32. *There exists an absolute constant C such that for any large K and M*

$$|F_{4KM}| \leq CK |F_K| \cdot |F_M|. \quad (7.1)$$

Proof. This will be a proof by greedy algorithm. First choose $y \in F_{4K}$ and consider needle N_y and triangles of certain size 3^{-j_y} , $j_y \leq N$ intersecting N_y . Consider any family of this sort having more than $4K$ elements. Fix such a family. We will "fathorize" it, i.e. we consider the father of each element in the family. Two things may happen: 1) there are more than $4K$ distinct fathers; 2) number of fathers is at most $4K$. In the latter case the number of fathers is at least $2K$. In fact, we slash the number of elements by fathorizing, but oit more than by factor of $1/2$. If the first case happens fathorize again, do this till we get to the second case.

After doing this procedure with all $x \in F_{4K}$ and all families of cardinality bigger than $4K$ of equal size triangles intersecting needle N_x we come to some awfully

complicated set of triangles. But we will consider now maximal-by-inclusion triangles of this family, the family of these maximal triangles is called \mathcal{F}_0 .

Choose triangle $Q_{00} \in \mathcal{F}_0$ such that its sidelength $\ell(Q_{00})$ is maximal possible in \mathcal{F}_0 . It is very important to notice that \mathcal{F}_0 contains at least $4K - 1$ triangles of the same size as Q_{00} pierced by a needle N_{y_0} . This is because of maximality of the lengthsize, the stack pierced by N_{y_0} could not be eaten up even partially by bigger in size triangles from some other stack. So let us call by $Q_{01}, \dots, Q_{02K-1}, \dots, Q_{0S}$, $S \geq 2K - 1$. They are of the same size as Q_{00} and all intersect a certain needle N_{y_0} .

Denote

$$I_0 = \text{proj } Q_{00}.$$

Consider all $q \in \mathcal{F}_0$ such that

$$\text{proj } q \cap 20 I_0 \neq \emptyset.$$

Call them $\mathcal{F}(Q_{00})$. Of course $\ell(q) \leq \ell(Q_{00})$. For every such q consider a Cantor square Q , $q \subset Q$, such that $\ell(Q) = \ell(Q_{00})$. Such Q 's form family $\tilde{\mathcal{F}}(Q_{00})$.

Lemma 33. *For every $y \in \mathbb{R}$ the needle N_y intersects at most $4K$ triangles of the family $\tilde{\mathcal{F}}(Q_{00})$.*

Proof. Suppose contrary. Then N_y intersects more than $4K$ of triangles from $\tilde{\mathcal{F}}(Q_{00})$. So $y \in F_{4K}$, and our pierced family is one of those which we considered at the begining. It can be fathorized. Then the square of size $\geq 2\ell(Q_{00})$ will be prrsent in \mathcal{F}_0 . Contradiction with maximality of length. □

Lemma 34. $\text{card } \tilde{\mathcal{F}}(Q_{00}) \leq 88K$.

Proof.

$$\begin{aligned} \text{card } \tilde{\mathcal{F}}(Q_{00}) \cdot \ell(Q_{00}) &= \sum_{Q \in \tilde{\mathcal{F}}(Q_{00})} \ell(Q) \leq \\ &\int_{22I_0} \text{card } \{Q \in \tilde{\mathcal{F}}(Q_{00}) : Q \cap N_y \neq \emptyset\} dy \leq \\ &4K \cdot 22\ell(Q_{00}). \end{aligned}$$

This is by Lemma 33. □

Lemma 35. *There exists an interval $J_0 \subset I_{y_0}$ such that $|J_0| \geq c \cdot |I_0|$ with a certain absolute positive c . And $J_0 \subset F_K$.*

Proof. We already noticed that $Q_{00}, Q_{01}, \dots, Q_{02K-1}$ intersect needle N_{y_0} . Then at least half of them have their center of symmetry to the right of N_{y_0} , or at least half of them have their center of symmetry to the left of N_{y_0} . Assume that the first case occurs. Then the segment $[y_0, c \cdot \ell(Q_{00})]$ obviously is contained in F_K . \square

Lemma 36. $|F_{4KM} \cap 20I_0| \leq C K \ell(Q_{00}) = C K |I_0|$.

Proof. Of course $F_{4KM} \subset F_{4K}$. For $y \in F_{4KM} \cap 20I_0$ the whole family of small triangles whose quantity is $> 4KM$ intersecting N_y will be inside one of those $Q \in \tilde{\mathcal{F}}(Q_{00})$, whose number is at most $88K$ by Lemma 34. Let us enumerate Q^1, \dots, Q^s , $s \leq 88K$ elements of $\tilde{\mathcal{F}}(Q_{00})$. So there exists $i = 1, \dots, s$ such that

$$y \in \text{dilated copy of } F_M \text{ in } \text{proj } Q^i.$$

Hence

$$F_{4KM} \cap 20I_0 \subset \cup_{i=1}^{88K} \text{dilated copy of } F_M \text{ in } \text{proj } Q^i.$$

So

$$|F_{4KM} \cap 20I_0| \leq \sum_{i=1}^{88K} \ell(Q^i) |F_M| \leq 88K \ell(Q_{00}) |F_M|.$$

\square

Lemma 37. $|F_{4KM} \cap 20I_0| \leq 88c^{-1} K |F_m| \cdot |J_0|$.

Now we want to repeat all steps for $F_{4K}^0 := F_{4K} \setminus 20I_0$. So we favorize triangles peirced by needles N_x , $x \in F_{4K}^0$. As before we get families \mathcal{F}_1 , maximal sidelength triangle Q_{11} , families $\mathcal{F}(Q_{11})$, $\tilde{\mathcal{F}}(Q_{11})$. Notice that $\mathcal{F}_1 < \mathcal{F}_0$ in the sense that for every $q \in \mathcal{F}_1$ there exists $q \in \mathcal{F}_0$ such that q is contained in Q . It is also clear that

$$\ell(Q_{11}) \leq \ell(Q_{00}).$$

Obviously Q_{00}, Q_{01}, \dots are not in \mathcal{F}_1 , their projections even do not intersect $\mathbb{R} \setminus 20I_0$.

There are at least $2K - 1$ brothers of Q_{11} : $Q_{12}, \dots, Q_{12K-1}, \dots$ in \mathcal{F}_1 such that they are of the same size $\ell(Q_{11})$ and they (and Q_{11}) intersect the same needle N_{y_1} , $y_1 \in \mathbb{R} \setminus 20I_0$. This is again the maximality of the sidelength among \mathcal{F}_1 triangles. Let $I_1 := \text{proj } Q_{11}$. Notice that

$$I_1 \cap I_0 = \emptyset.$$

In fact, $y_1 \in I_1, y_1 \notin 20I_0$, Q_{11} size is much smaller than $20|I_0|$. We consider all $q \in \mathcal{F}_1$ such that

$$\text{proj } q \cap (20I_1 \setminus 20I_0) \neq \emptyset.$$

Call this family $\mathcal{F}(Q_{11})$. For every $q \in \mathcal{F}(Q_{11})$ consider Cantor triangle Q containing q and of the size $\ell_1 = \ell(Q_{11})$. Maximal-by-inclusion among such Q 's form $\tilde{\mathcal{F}}(Q_{11})$.

Lemma 38. *For any $y \in R \setminus 20I_0$, N_y intersects at most $4K$ triangles of $\tilde{\mathcal{F}}(Q_{11})$.*

Proof. Suppose contrary. Then there exists $y'_1 \in F_{4K} \cap (\mathbb{R} \setminus 20I_0)$, and a subfamily of $\tilde{\mathcal{F}}(Q_{11})$ of cardinality bigger than $4K$ intersects $N_{y'_1}$. It can be fathorized. Then triangles of size $\geq 2\ell(Q_{11})$ would belong to \mathcal{F}_1 . This contradicts the maximality of $\ell(Q_{11})$. □

Lemma 39. *For any $z \in \mathbb{R}$, N_z intersects at most $8K$ triangles of $\tilde{\mathcal{F}}(Q_{11})$.*

Proof. Suppose contrary. Then there exists $z \in F_{4K}$, and a subfamily of $\tilde{\mathcal{F}}(Q_{11})$ of cardinality bigger than $4K$ intersects N_z . Now there is an end-point of $20I_1 \setminus 20I_0$ (call it a), which is closest to z . Let it be on the right of z . Then another end-point is also on the right but farther away. As every triangle from the family has a) z in its projection, and b) a certain point to the right of a in its projection (their projections intersect $20I_1 \setminus 20I_0$ —by definition), then all of them have a in its projection. Let us be lavish and say that 50 percent of them have a in their projection (the fact is that it is not lavishness, it is necessity: next step will be to consider in the future $20I_2 \setminus (20I_0 \cup 20I_1)$, and there can be 2 closest points to z : one on the left, say, b , and one on the right, say, a , and we can guarantee that 50 percent of our triangles have either b or a in their projections simultaneously). We use the previous Lemma 38, and get that this 5) percent is $\leq 4K$. So we are done. □

Lemma 40. $\text{card } \tilde{\mathcal{F}}(Q_{11}) \leq 172K$.

Proof.

$$\begin{aligned} \text{card } \tilde{\mathcal{F}}(Q_{11}) \cdot \ell(Q_{11}) &= \sum_{Q \in \tilde{\mathcal{F}}(Q_{11})} \ell(Q) \leq \\ &\int_{22I_1} \text{card } \{Q \in \tilde{\mathcal{F}}(Q_{11}) : Q \cap N_y \neq \emptyset\} dy \leq \end{aligned}$$

$$8K \cdot 22\ell(Q_{11}).$$

This is by Lemma 33. □

Lemma 41. *There exists an interval $J_1 \subset I_1$, $|J_1| \leq c \cdot |I_1|$, such that $J_1 \subset F_K$.*

Proof. The same proof as for Lemma 35. □

Lemma 42. $|F_{4KM}^0 \cap 20I_1| \leq C K \ell(Q_{11} \leq C|, K |I_1|$.

Proof. The same proof as for Lemma 36. □

Combining Lemmas 41, 42 we get

Lemma 43. $|F_{4KM}^0 \cap 20I_1| \leq C c^{-1} K |J_1|$.

We continue by introducing

$$F_{4KM}^1 = F_{4KM} \setminus (20I_0 \cup 20I_1).$$

We repeat the whole procedure. There will be $I_2, J_2 \subset I_2 \cap F_K, |J_2| \geq c \cdot |I_2|$:

$$I_2 \cap (I_1 \cup I_0) = \emptyset,$$

$$|F_{4KM} \cap 20I_2| \leq C c^{-1} K |J_2| |F_M|,$$

et cetera.

Finally,

$$|F_{4KM}| \leq |F_{4KM} \cap 20I_0| + |(F_{4KM} \setminus 20I_0) \cap 20I_1| + \dots + |(F_{4KM} \setminus 20I_0 \cup 20I_1 \cup \dots \cup 20I_{j-1}) \cap 20I_j| + \dots \leq$$

$$C' K |F_M| \sum_{j=0}^{\infty} |J_j| \leq C' K |F_M| |F_K|.$$

We are done with Theorem 32. □

Now we can prove Theorem 31.

Proof. Let $E_j := \{x : f_{n,\theta}(x) > (4K)^{j+1}\}$, $j = 0, 1, \dots$. We know by Theorem 32 that

$$|E_j| \leq (CK)^j |E_0|^{j+1}.$$

Hence,

$$\int f_{n,\theta}(x)^2 dx \leq 4K \int f_{n,\theta}(x) dx + \sum_{j=0}^{\infty} \int_{E_j \setminus E_{j+1}} f_{n,\theta}(x)^2 dx \leq$$

$$4CK + (4K)^{j+2} (CK)^j |E_0|^{j+1}.$$

If $|\{x : f_N^*(x) > K\}| \leq 1/K^{2+\tau}$ then for all $n \leq N$ we can immediately read the previous inequality as

$$\int f_{n,\theta}(x)^2 dx \leq C(\tau) K.$$

□

8. DISCUSSION

8.1. Difficulties for more general self-similar sets. Analytic tiling in every direction is unique to the gasket, though perhaps there is some hope that something similar occurs for typical directions in the arbitrary case. Suppose we had 5 self-similarities, and that for some direction θ , we had $\phi_\theta(x_0) = 1 + (-i) + i + e^{2\pi i/3} + e^{4\pi i/3} = 0$. Then clearly, taking fifth powers of the summands results in another zero with exactly the same summands, in complete and utter contrast to the three-point case. Similar examples using partitions into relatively prime roots of unity exist for numbers other than 5.

At any rate, our arguments without analytic tiling can still get the estimate $Fav(\mathcal{G}_n) \leq e^{-c\sqrt{\log n}}$. However, we cannot get anything at all in the general case if we cannot find a suitable substitute for the Riesz estimates of Lemma 8. Perhaps some other kind of analytic tiling can resolve one or both issues.

8.2. Explicit exponents, and an estimate for degenerate gaskets. Fix two self-similarity centers p_1, p_3 with $|p_1 - p_3| = 1$ and choose a third self-similarity center p_2 so that $|p_2 - p_3|, |p_2 - p_1| \leq 1$. Define the **degeneracy** δ of this configuration (and the resulting gasket) to be twice the area of the triangle with corners p_j . In particular, if one fixes $p_1 = 0, p_3 = 1, p_2 = 1/2 + 2i\delta$, then G approaches a line segment as δ approaches 0, and in general, any upper bound on $Fav(\mathcal{G}_n)$ should break down as $\delta \rightarrow 0$. We will show this by highlighting the places where δ makes a difference.

For $j = 1, 2, 3$, write $p_j = re^{i\theta_j}$. This can be done with r not depending on j , since three non-colinear points define a circle. For $\theta \in [0, 2\pi]$, let $c_j := r\cos(\theta_j - \theta)$, and let $s_j := r\sin(\theta_j - \theta)$. One gets

$$\phi_\theta(x) = \frac{1}{3} \sum_{j=1}^3 e^{-ic_j x}$$

$$\varphi_t(\tilde{x}) = e^{-i\frac{c_1}{c_3 - c_1} \tilde{x}} (1 + e^{-it\tilde{x}} + e^{-i\tilde{x}}),$$

where $\tilde{x} = (c_3 - c_1)x$ and $t = \frac{c_2 - c_1}{c_3 - c_1}$. We can consider this with the indices permuted, so that we are always in the case $c_1 \leq c_2 \leq c_3$. Thus

$$\delta \|\phi_\theta\|_{L^2(x)} \leq \|\varphi_t\|_{L^2(\tilde{x})} \leq \|\phi_\theta\|_{L^2(x)}.$$

Lemma 27 gains a δ^{-1} on the right-hand side wherever it is applied, since the frequencies might be packed in a lot tighter. Thus Proposition 5 is the same

$$\int_I |P_{1,t}(x)|^2 dx \geq C3^m,$$

but it is not true unless $CK/\delta \leq 3^{3m/4}$ for all $N \geq N^*$. So $N^* \geq C\delta^{-1}$.

Propositions 7 also picks up a δ^{-1} on the right hand side, since P_1^\sharp used Lemma 27 as well.

The final estimate becomes

$$\int_{\tilde{E}} \int_{SSV(t)} |P_{1,t}(x)|^2 dx dt \leq 3^{Bm} C m K 3^m \cdot C'' m 3^m \left(\frac{7}{9}\right)^\ell \cdot \delta^{-1},$$

so the $(\frac{7}{9})^\ell$ term has to work that much harder. But easily the right hand side is at most $3^{m/4} \delta^{-1} \leq 3^{m/2}$,

which again is no trouble if $N^* \geq \delta^{-2}$.

Now we need to deal with the change of variables $\theta \rightarrow t$.

Lemma 44.

$$\delta \leq \left| \frac{dt}{d\theta} \right| \leq \delta^{-1}$$

Proof. Remember, we are working in a relabeling where $c_1 \leq c_2 \leq c_3$

$$\frac{dt}{d\theta} = \frac{(c_3 - c_1)(s_2 - s_1) - (c_2 - c_1)(s_3 - s_1)}{(c_3 - c_1)^2}.$$

Note that the numerator is constant⁷. So if you evaluate this constant when $c_1 = c_2$, the surviving term is exactly δ , the base times the height of the triangle. The result follows, since $\delta^2 \leq (c_3 - c_1)^2 \leq 1$ □

Let

$$f_{N,\theta}^*(s) := \sup_{n \leq N} f_{n,\theta}(s)$$

$$A_{K,\theta}^* := \{s : f_{N,\theta}^*(s) \geq K\}$$

⁷Differentiate without multiplying anything out

$$E_\theta := \{\theta : |A_{K,\theta}^*| \leq \frac{1}{K^3}\}.$$

Thus $|E_\theta| \leq |E|\delta^{-1} \leq \frac{1}{\delta K}$.

Putting everything together with Section 3, we get

$$Fav(\mathcal{G}_{NK^3}) \leq \text{average length for good angles} + 3|\text{bad angles}| \leq \frac{C}{K} + \frac{3}{\delta K}$$

$N = K^{1/\epsilon_0}$, so

$$Fav(\mathcal{G}_{K^{3+1/\epsilon_0}}) \leq \frac{C}{\delta K}, \text{ or}$$

$$Fav(\mathcal{G}_M) \leq \frac{C}{\delta M^{\epsilon_0/(3\epsilon_0+1)}},$$

so long as $M \geq \delta^{-2(1+3\epsilon_0)}$. Otherwise, we have the upper bound 3, and the bound is valid for all cases. So for all n , we can write

$$Fav(\mathcal{G}_n) \leq \frac{C_{\epsilon_0}}{\delta n^{\epsilon_0/(1+3\epsilon_0)}}$$

Note that $\epsilon_0 < 1/11$ is sufficient for all considerations. In fact, one can improve this: $m = 2\epsilon_0 \log N$ was a bit gratuitous; if one uses $m = (1 + \eta)\epsilon_0$, one sees that in fact $\epsilon_0 < 1/6$ is sufficient. This can be seen by working from the beginning of Section 2 up through equation (2.6) with this remark in mind. Therefore, one may use any $p < 1/9$. Also a bit gratuitous is the exponent 3 in Theorem 31. Since $2 + \eta$, $\eta > 0$, could have been used instead, $p < 1/8$ could also have been used in Theorem 1. It seems not so easy to improve anything else any further at this point.

REFERENCES

1. M. Bateman, A. Volberg, *An estimate from below for the Buffon needle probability of the four-corner Cantor set*, arXiv:math. 0807.2953v1, 2008, pp. 1-11.
2. M. Bateman, N. Katz, *Keakeya sets in Cantor directions*, arXiv:math. 0609187v1, 2006, pp. 1-10.
3. M. Bateman, *Keakeya sets and the directional maximal operators in the plane*, arXiv:math.CA 0703559v1, 2007, pp. 1-20.
4. A. S. Besicovitch, *Tangential properties of sets and arcs of infinite linear measure*, *Bull. Amer. Math. Soc.* **66** (1960), 353-359.
5. M. BOND, A. VOLBERG: *Estimates from below of the Buffon noodle probability for undercooked noodles*, arXiv:math/0811.1302v1, 2008, pp. 1-10.
6. K. J. Falconer, *The geometry of fractal sets*. Cambridge Tracts in Mathematics, 85. C.U.P., Cambridge-New York, (1986).

7. U. Keich, On L^p bounds for Keakeya maximal functions and the Minkowski dimension in \mathbb{R}^2 , *Bull. London. Math. Soc.* **31** (1999), pp. 213–221.
8. R. Kenyon, *Projecting the one-dimensional Sierpinski gasket*, *Israel J. Math.* **97** (1997), 221–238.
9. I. Laba, K. Zhai, *Favard length of product Cantor sets*, arXiv:0902:0964v1, Feb. 5 2009.
10. J. C. Lagarias and Y. Wang, *Tiling the line with translates of one tile*, *Invent. Math.* **124** (1996), 341–365.
11. P. Mattila, *Orthogonal projections, Riesz capacities and Minkowski content*, *Indiana Univ. Math. J.* **39** (1990), 185–198.
12. P. Mattila, *Hausdorff dimension, projections, and the Fourier transform*, *Publ. Mat.*, **48** (2004), pp. 3–48.
13. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.
14. F. Nazarov, Y. Peres, A. Volberg *The power law for the Buffon needle probability of the four-corner Cantor set*, arXiv:0801.2942, 2008, pp. 1–15.
15. Y. Peres, K. Simon and B. Solomyak, *Self-similar sets of zero Hausdorff measure and positive packing measure*, *Israel J. Math.* **117** (2000), 353–379.
16. Y. Peres and B. Solomyak, *How likely is Buffon’s needle to fall near a planar Cantor set?* *Pacific J. Math.* **204**, 2 (2002), 473–496.
17. I. J. Schoenberg, *On the Besicovitch-Perron solution of the Keakeya problem*, *Studies in mathematical analysis and related topics*,
18. T. Tao, *A quantitative version of the Besicovitch projection theorem via multiscale analysis*, pp. 1–28, arXiv:0706.2446v1 [math.CA] 18 Jun 2007.

MATTHEW BOND, DEPT. OF MATH., MICHIGAN STATE UNIVERSITY. bondmatt@msu.edu

ALEXANDER VOLBERG, DEPT. OF MATH., MICHIGAN STATE UNIV. AND THE UNIVERSITY OF EDINBURGH. volberg@math.msu.edu