THE POWER LAW FOR BUFFON NEEDLE LANDING NEAR SIERPINSKI GASKET

Abstract. In this paper we modify the method of Nazarov, Peres, and Volberg [15] to get a power estimate from above of the Buffon needle probability of the nth partially constructed Sierpinski gasket of Hausdorff dimension 1.

1. Introduction

Among self-similar planar sets of Hausdorff dimension 1, the simplest are the Sierpinski gasket $\mathcal{G}$ (formed by three self-similarities) and the square 1/4 corner Cantor set $\mathcal{K}$ (formed by four self-similarities). By the Besicovitch projection theorem, these irregular sets of positive and finite Hausdorff H^1 measure must have zero length in almost every orthogonal projection onto a line. One may partially construct these sets in the usual way by taking the convex hull and then taking the union of all possible images of n-fold compositions of the similarity maps. Then one may ask the rate at which the Favard length – the average over all directions of the length of the orthogonal projection onto a line in that direction – of these sets $\mathcal{G}_n$ and $\mathcal{K}_n$ decay to zero as a function of n.

First quantitative results were obtained in [17],[19]; in the latter paper a general way of making a quantitative statement from the Besicovitch theorem is considered. But being rather general this method does not give a good estimate for self-similar structures such as $\mathcal{K}_n$ or $\mathcal{G}_n$. In [15] a power upper bound was obtained for the 1/4 Cantor set.

The lower bound was obtained relatively easily in a paper of Bateman and Volberg [5] (see also [6] for a related question): it is $c\frac{\log n}{n}$. The argument painlessly yields the same lower bound for $\mathcal{G}_n$.

The upper bound for $\mathcal{K}_n$ is due to Nazarov, Peres, and Volberg [15]: if $p < 1/6$, $Fav(\mathcal{K}_n) \leq \frac{c_p}{n^p}$. To get this estimate, the radial symmetry was used in addition to a reflection symmetry which $\mathcal{G}_n$ lacks. The main idea of [15] of separation the

1991 *Mathematics Subject Classification.* Primary: 28A80. Fractals, Secondary: 28A75, Length, area, volume, other geometric measure theory 60D05, Geometric probability, stochastic geometry, random sets 28A78 Hausdorff and packing measures.

Research of the authors was supported in part by NSF grants DMS-0501067, 0758552 .

directions into good ones and singular ones holds for $\mathcal{G}_n$, but the changes which must be made are not completely superficial. The goal of this paper is to make whatever changes are necessary to find some upper bound of the decay rate for the n-th partial Sierpinski gasket. The struggle, as often in analysis, is with the set of small values of a certain function (in our case the function is an exponential polynomial). In [15] this exponential polynomial happened to be just a sine function. The case of the gasket is much closer to the generic case as the polynomial becomes a rather general 3-term exponential sum. Notice that, in fact, it is an entire function of 2 variables: one variable is given by the choice of the direction of projection (and in our considerations below should be made even complex by some reason!), another variable is its "spectral" variable. Sorting out the zeros and the set of small values of this entire function will give us some headache. However, the advantage is that the Sierpinski gasket provides a much better glimpse at the general self-similar sets completely irregular in the sense of Besicovitch than the 1/4 corner Cantor set. We believe that using this approach one can work with all such sets. We pay for that: while [15] combined combinatorics with Fourier analyis, here we need to add a certain amount of complex analysis into reasoning. Rather strangely, a claim in the spirit of the Carleson Embedding Theorem, in the form of Lemma 26, plays an important part in our reasoning.

Notice that product structure of the underlying Cantor set was recently explored in Laba-Zhai's paper [10], where they extended the result of [15] to product Cantor sets. Their argument involves a combinatorial reasoning related to tiling studied by Kenyon [9] and Lagarias-Wang [11].

Already in [5] we got rid of the product structure and analized the zeros of our exponential polynomials. One of the main idea of the present paper, and its main diffrerence with [15], [10] and [5] is that we introduce the notion of analytic tiling. We see it as follows: even if the tiling does not exist, a ceratin shadow of it cast over the Fourier side, still can persist. And this is what happens and what helps to obtain the power estimate, which is a couple of logarithms better than the estimate in [5]. The shadow of tiling we mentioned above consists, roughly speaking, in the fact that the small discs of correct size centered at the zeros of Fourier tarnsform of "Cantor" measure cover the set of small values ("small" is also correct) of this Fourier transform.

Consider the function $f_{n,\theta} : \mathbb{R} \to \mathbb{N}$ defined by

$$f_{n,\theta} = \sum_{\text{Sierpinski triangles T}} \chi_{proj_\theta(T)}$$

Note that $Fav(\mathcal{G}_n) = \pi^{-1} \int_0^\pi |supp(f_{n,\theta})| d\theta$. In [15] and [5], the L^p norms of the analog of this function for squares were studied to obtain Buffon needle probability estimates for $\mathcal{K}_n$ – in [5], $p = 1, 2$ were related to $\chi_{supp(f_{n,\theta})}$ via the Cauchy inequality, while in [15], $p = 2$ was studied via Fourier transforms and related to the case of $p = \infty$. Indeed, if we ignore the averaging over θ for the moment and consider a sum of characteristic functions of intervals whose L^1 norm is 1, then heuristically, the argument is that as the mass becomes more concentrated on smaller sets, the L^p norms will grow. Thus for $p > 1$, a large L^p norm should indicate that the support of a function is small, and vice versa. Therefore, to show that the $Fav(\mathcal{G}_n)$ is small, we will show that if we fix N large, then for most angles θ it will follow that $||f_{n,\theta}||_\infty$ is large for at least one $n < N$.

The main result of this article is the following estimate (of course far from being optimal, see Section 6.1 for the further discussion).

Theorem 1.

$$Fav(\mathcal{G}_n) \leq \frac{C}{n^c}\,.$$

2. The Fourier-analytic part

Our computations will be simplified if we first rescale $\mathcal{G}_n$ by a factor absolutely comparable to 1 and bound the triangles by discs and study this set instead. That is, for $\alpha \in \{-1, 0, 1\}^{n+1}$ let

$$z_\alpha := \sum_{k=0}^{n} (\frac{1}{3})^k e^{i\pi[\frac{1}{2}+\frac{2}{3}\alpha_k]},$$

and then let

$$\mathcal{G}_n := \bigcup_{\alpha \in \{-1,0,1\}^{n+1}} B(z_\alpha, 3^{-n}).$$

Note that $\mathcal{G}_n$ has 3^{n+1} discs of radius 3^{-n}. After a rescaling, the usual $n + 1$st Sierpinski gasket (composed of 3^{n+1} triangles) sits inside of $\mathcal{G}_n$. We may still speak of the approximating discs as "Sierpinski triangles."

Observe that $f_{n,\theta} = \nu_n * 3^n \chi_{[-3^{-n}, 3^{-n}]}$, where $\nu_n := *_{k=0}^{n} \widetilde{\nu}_k$ and

$$\widetilde{\nu}_k = \frac{1}{3}[\delta_{3^{-k} cos(\pi/2-\theta)} + \delta_{3^{-k} cos(-\pi/6-\theta)} + \delta_{3^{-k} cos(7\pi/6-\theta)}].$$

Let us fix K and let

$$E := E_K := \{\theta : |\{x : f_N^*(x) := \sup_{n \le N} f_{n,\theta}(x) \le K\}| \ge \frac{1}{K^3}\} .$$

Let $K >> 1$, and suppose $|E| := |E_K| \ge \frac{1}{K}$. We will show that if $K = N^{\epsilon_0}$ (ϵ_0 is a small absolute positive number), this would bring us the contradiction. Therefore, we will get an estimate from above on the measure of the set E of "bad" directions.

One of the most important part will be played by the following theorem proved in Section 7.

Theorem 2. *Let $\theta \in \{\theta : |\{x : f_N^*(x) := \sup_{n \le N} f_{n,\theta}(x) \le K\}| \le \frac{1}{K^3}\}$. Then*

$$\max_{0 \le n \le N} \|f_{n,\theta}\|^2_{L^2(\mathbb{R})} \le C\, K .$$

Let $N = K^{1/\epsilon_0}$ (whenever an integer is defined to be a non-integer, it is understood that one rounds). Then $\forall \theta \in E_K$,

$$K \ge ||f_{N,\theta}||^2_{L^2(x)} \approx ||\widehat{f_{N,\theta}}||^2_{L^2(y)} \ge C \int_1^{3^{N/2}} |\widehat{\nu}(y)|^2 dy$$

Splitting $[1, 3^{N/2}]$ into $N/2$ pieces $[3^k, 3^{k+1}]$ and taking blocks of such consecutive pieces, the blocks cannot all have large intergrals simultaneously. That is, if we fix $0 < A' < B' < 1/2$, then $\forall m \in [0, A'N] \exists n \in [B'N, N/2]$ s.t.

$$\frac{1}{|E_K|} \int_{E_K} \int_{3^{n-m}}^{3^n} |\widehat{\nu_N}|^2 dy d\theta \le CKm/N .$$

So if

$$E := \{\theta \in E_K : \int_{3^{n-m}}^{3^n} |\widehat{\nu_N}|^2 dy d\theta \le 2CKm/N\} .$$

then $|E| \ge \frac{1}{2S^2}$.

Define $c_1 = cos(\theta - \pi/2)$, $c_2 = cos(\theta - 7\pi/6)$, $c_3 = cos(\theta + \pi/6)$, and similarly, $s_1 = sin(\theta - \pi/2)$, etc. Let

$$\phi_\theta(y) = \frac{1}{3} \sum_{j=1}^{3} e^{-ic_j y}.$$

Then $\widehat{\nu_N}(y) = \prod_{k=0}^{N} \phi_\theta(3^{-k}y) \approx \prod_{k=0}^{n} \phi_\theta(3^{-k}y)$ for $y \in [3^{n-m}, 3^n]$. So changing variable ($y \to 3^n y$) and reindexing the product ($k \to n-k$), we get

$$\int_{3^{n-m}}^{3^n} |\widehat{\nu_N}|^2 dy d\theta \approx \int_{3^{n-m}}^{3^n} \prod_{k=0}^{n} |\phi_\theta(3^{-k}y)|^2 dy d\theta = 3^n \int_{3^{-m}}^{1} \prod_{k=0}^{n} |\phi_\theta(3^k y)|^2 dy d\theta$$

So for $\theta \in E$, $3^n \int_{3^{-m}}^{1} \prod_{k=0}^{n} |\phi_\theta(3^k y)|^2 dy d\theta \leq \frac{CKm}{N}$. Later, we will let $m \approx \log K$ and $l = \alpha \log K$ (for an appropriate α) and show that $\exists \theta \in E$ such that

$$\int_{3^{-m}}^{1} \prod_{k=0}^{n} |\phi_\theta(3^k y)|^2 dy \geq C 3^{-n+m-A\,\ell}, \tag{2.1}$$

resulting in a choice of m.

First, let us write $\prod_{k=0}^{n} \phi_\theta(3^k y) = P_\theta(y) = P_{1,\theta}(y) P_{2,\theta}(y)$, where

$$P_{1,\theta}(y) = \prod_{k=0}^{m} \phi_\theta(3^k y) \text{ and } P_{2,\theta}(y) = \prod_{k=m+1}^{n} \phi_\theta(3^k y).$$

We want

$$\int_{3^{-m}}^{1} |P_{2,\theta}|^2 dy \geq C 3^{m-n} \tag{2.2}$$

with a proportion of the contribution to the integral separated away from the complex zeroes of $P_{1,\theta}$.

First, Salem's trick for $\int_0^1 |P_{2,\theta}(y)|^2 dy$:

Let $h(y) := (1 - |y|)\chi_{[-1,1]}(y)$, and note that $\hat{h}(\lambda) = C\frac{1-cos\lambda}{\lambda^2} > 0$. Then if we write $P_{2,\theta} = 3^{m-n} \sum_{j=1}^{3^{n-m}} e^{i\lambda_j y}$, we get

$$\int_0^1 |P_{2,\theta}(y)|^2 dy \geq 2 \int_{-1}^{1} h(y) |P_{2,\theta}|^2 dy \approx (3^{m-n})^2 [3^{n-m} + \sum_{j \neq k, j,k=1}^{3^{n-m}} \hat{h}(\lambda_j - \lambda_k)] \geq 3^{m-n}.$$

To show that this is not concentrated on $[0, 3^{-m}]$, we will use Lemma 26. We get

$$\int_0^{3^{-m}} |P_{2,\theta}(y)|^2 dy = 3^{-m} \int_0^1 |P_{2,\theta}(3^{-m}y)|^2 dy = 3^{-m}(3^{m-n})^2 \int_0^1 |\sum_{j=1}^{3^{n-m}} e^{i\lambda_j 3^{-m} y}|^2 dy.$$

Note that in this expression, the frequencies $\beta_j := 3^{-m}\lambda_j$ are the frequencies $\mu \in \Lambda_{n-m}$ of $\widehat{f_{n-m,\theta}}$, after they have been subjected to two changes of variables acting on y by a cumulative factor of 3^{n-m}: $\beta_j = 3^{-m}\lambda_j = 3^{n-m}\mu$'s. By the definition of E_K and by Theorem 2, β_j satisfy the following

$$\int_{\mathbb{R}} (\sum_j \chi_{[\beta_j - 1, \beta_j + 1]}(x))^2 \, dx \leq K \, 3^{n-m}\,. \tag{2.3}$$

In fact, by the definition of E_K and by Theorem 2 we have

$$\int_{\mathbb{R}} \Big(\sum_{\mu \in \Lambda_{n-m}} \chi_{[\mu - 3^{-(n-m)}, \mu + 3^{-(n-m)}]}(s) \Big)^2 \, ds \leq C\, K\,. \tag{2.4}$$

Changing variable $s = 3^{-(n-m)} x$ we come from (2.4) to (2.3) as β's $= 3^{n-m}\mu$'s. So the Lemma (26) tells us that

$$\int_0^{3^{-m}} |P_{2,\theta}|^2 dy \leq C 3^{-m} (3^{m-n})^2 K 3^{n-m} \leq C \frac{3^{-m} K}{3^{n-m}} \leq \frac{1}{2} \int_0^1 |P_{2,\theta}(y)|^2 dy,$$

if we introduce the assumption $3^m = C' K$ for C' large enough. So now we have 2.2

2.1. **The estimate of $P_{2,\theta}$ on the set of small values of $P_{1,\theta}$.** To get 2.1 from 2.2, we will show that a proportion of 2.2 must have come from outside of the **set of small values** $SSV(\theta, \ell)$ of $P_{1,\theta}$, so that in 2.2, we may restrict the integration domain to the complement of $SSV(\theta, \ell)$ and bound P_1 by $3^{-A\ell}$ from below.

Let $\ell = C_0\, m$, where the large absolute C_0 will be chosen later.

Definition.

$$SSV(\theta, \ell) := \{ y \in [0,1] : |P_{1,\theta}(y)| < 3^{-A\,\ell} \}\,,$$

where A is another large absolute constant to be seen in Section 4.

This is the desired inequality:

$$\frac{1}{|E|} \int_E \int_{[3^{-m},1] \cap SSV(\theta,\ell)} |P_{2,\theta}(y)|^2 \, dy\, d\theta \leq \frac{\varepsilon_0}{3^{n-m}} \leq 0.5\, \frac{1}{|E|} \int_E \int_{[3^{-m},1]} |P_{2,\theta}(y)|^2 \, dy\, d\theta, \tag{2.5}$$

because it gives us

$$\frac{1}{|E|} \int_E \int_{[3^{-m},1] \setminus SSV(\theta,\ell)} |P_{2,\theta}(y)|^2 \, dy\, d\theta \geq 0.5\, \frac{1}{|E|} \int_E \int_{[3^{-m},1]} |P_{2,\theta}(y)|^2 \, dy\, d\theta \geq a\, \frac{1}{3^{n-m}}\,.$$

To get it, we will need to split $P_{2,\theta}(y)$ into two parts, $P^\sharp_{2,\theta}(y)$ and $P^\flat_{2,\theta}(y)$, because Lemma 26 applied to $P^\sharp_{2,\theta}(y)$ will get us part of the way there, and the claims of Section 4 applied to $P^\flat_{2,\theta}(y)$ will finish the estimate.

Introduce

$$P^\flat_{2,\theta}(y) := \prod_{k=m+1}^{m+\ell} \phi_\theta(3^k y)\,,\; P^\sharp_{2,\theta}(y) := \prod_{k=m+\ell+1}^{n} \phi_\theta(3^k y)\,.$$

Also

$$R(x) := \prod_{k=m+1}^{m+\ell} r(3^k\, x) := \prod_{k=m+1}^{m+\ell} \frac{7+2\cos(3^k x)}{9}\,.$$

We will prove now

Lemma 3. $|P^\flat_{2,\theta}(y)|^2 \le \min(R(y(c_2(\theta)-c_1(\theta))), R(y(c_3(\theta)-c_1(\theta))))\,.$

Proof. Notice that

$$|2\cos\alpha + e^{i\beta}|^2 \le 7 + 2\cos(\alpha-\beta)\,.$$

In fact,

$$|2\cos\alpha + e^{i\beta}|^2 = 4\cos^2\alpha + 1 + 4\cos\alpha\cos\beta \le 5 + 2\cos(\alpha+\beta) + 2\cos(\alpha-\beta) \le$$

$$7 + 2\cos(\alpha-\beta)\,.$$

We can write

$$|e^{ic_1y} + e^{ic_2y} + e^{ic_3y}|^2 = |1 + e^{i(c_2-c_1)y} + e^{i(c_3-c_1)y}|^2 =: |1 + e^{it_1} + e^{it_2}|^2\,,$$

where $t_1 := (c_2-c_1)y, t_2 := (c_3-c_1)y$. This is the same as $|e^{-i\frac{t_1+t_2}{2}} + 2\cos(\frac{t_1-t_2}{2})|^2$. We use $\alpha = \frac{t_1-t_2}{2}, \beta = -\frac{t_1+t_2}{2}$, then $\alpha-\beta = t_1 = (c_2-c_1)y$. By symmetry we could have used $\alpha+\beta = -t_2 = -(c_3-c_1)y$.

□

In Section 4, we will see that SSV is contained in a union of small neighborhoods of the complex zeroes of $P_{1,\theta}$, and that the zeroes are in fact simple, depending differentiably on θ. So we divide SSV into the intersections of the neighborhoods of these zeroes with the real interval $[3^{-m}, 1]$. Lemma 24 says that within one such interval, our Riesz estimates on $P^\flat_{2,\theta}(y)$ are absolutely comparable independent of y.

In the following, we use notations of subsection 4.9: $j(i,t,\theta)$ will be a small interval containing $y(i,t,\theta)$, here points μ_i's enumerate zeros of $\Phi_m(z) = \prod_{s=0}^m \phi_\theta(3^{-s}z))$. Point $y(i,t,\theta)$ will be the real part of a complex zero of $P_{1,\theta} = \prod_{s=0}^m \phi_\theta(3^s z))$. At any rate, the intervals $j(i,t,\theta)$ united over $i = 1, ..., C\,3^m$ give $SSV(\theta,\ell)$. (See Section 4 for details.)

By indexing all zeros of $\Phi_m(z) := \prod_{s=0}^m \phi(3^{-s}z)$ in $[0, 3^m] \times [-\delta_0, \delta_0]$ bu μ_i as in Section 4, we write (see the notations in in subsection 4.9)

$$3^{-m} g_1(i,t,\theta) = y(i,t,\theta)(c_2(\theta)-c_1(\theta))\,,$$

and

$$3^{-m} g_2(i,t,\theta) = y(i,t,\theta)(c_3(\theta) - c_1(\theta))\,.$$

Later, we will see that there are a few pathological directions $\theta \in W$, which we will isolate in a small neighborhood W_m:

$$W := \{\pi/2, 5\pi/6, \pi/6\}. \text{ Let } W_m := \cup_{w\in W} B(w, 2\cdot 3^{-40m}).$$

Now we are going to estimate

$$I := \frac{1}{|E|}\int_E \int_{[0,1]\cap SSV(\theta,\ell)} |P^\flat_{2,\theta}(y)|^2 |P^\sharp_{2,\theta}(y)|^2\, dy\, d\theta\,.$$

Recalling that $\frac{1}{|E|} = S^2$ we have

$$I \le S^2 \int_{2W_m} \int_{[0,1]\cap SSV(\theta,\ell)} |P^\sharp_{2,\theta}(y)(y)|^2\, dy\, d\theta +$$

$$C_4\, S^2 \sum_{t=1}^{T} \int_{J^t\cap E} |P^\flat_{2,\theta}(y)|^2 \int_{[0,1]\cap SSV(\theta,\ell)} |P^\sharp_{2,\theta}(y)|^2\, dy\, d\theta =: I_W + \sum_t I_t\,.$$

Now, $SSV(\theta,\ell) \subset \cup_{k=1}^{K(t)} \cup_{s=\kappa}^{m} j(s,k,t,\theta)$ (Lemma 23).

Let us first estimate

$$\int_{j(i,t,\theta)} |P^\sharp_{2,\theta}(y)|^2 dy\,, \text{ having in mind that } \theta \in E\,.$$

Using that $\theta \in E$ and applying the Lemma 26 we get (recall that the length of $j(i,t,\theta)$ is $3^{-m-\ell}$ and $|c_j| = 1$ appears because of the change of variable $[0,1] \to j(i,t,\theta)$, $y = 3^{-m-\ell}x + b$, $c_j = e^{\lambda'_j b}$)

$$\int_{j(i,t,\theta)} |P^\sharp_{2,\theta}(y)|^2 dy = (3^{m+\ell-n})^2 \int_{j(i,t,\theta)} |\sum_{j=1}^{3^{n-m-\ell}} e^{i\lambda'_j y}|^2 dy\,.$$

Notice that the frequencies $3^{-m-\ell}\lambda'_j$ here are $3^n \cdot 3^{-m-\ell}$ multiples of frequencies of $\hat f_{n-m-\ell}$. Call those latter μ's. The set of them is called $\Lambda_{n-m-\ell}$. We know by the definition of $E = E_K$ and Theorem 2 that

$$\int_{\mathbb{R}} (\sum_{\mu\in\Lambda_{n-m-\ell}} \chi_{[\mu-3^{-n+m+\ell}, \mu+3^{-n+m+\ell}]}(s))^2\, ds \le C\,K\,. \tag{2.6}$$

Changing variable $s = 3^{-n+m+\ell}x$ one gets

$$\int_{\mathbb{R}} (\sum_{j=1}^{3^{n-m-\ell}} \chi_{[\lambda'_j-1, \lambda'_j+1]}(x))^2\, dx \le C\,K\,3^{n-m-\ell}\,. \tag{2.7}$$

Using Lemma 26 we get

$$\int_0^1 |\sum_{j=1}^{3^{n-m-\ell}} e^{i3^{-m-\ell}\lambda'_j y}|^2\, dy \le C\, K\, 3^{n-m-\ell}\,.$$

So

$$\int_{j(i,t,\theta)} |P^\sharp_{2,\theta}(y)|^2 dy \le (3^{m+\ell-n})^2 3^{-m-\ell} \int_0^1 |\sum_{j=1}^{3^{n-m-\ell}} e^{i3^{-m-\ell}\lambda'_j y}|^2\, dy \le$$

$$C\, K\, (3^{m+\ell-n})^2 3^{-m-\ell} 3^{n-m-\ell} = \frac{C\, 3^{-m}\, K}{3^{n-m}}\,.$$

Therefore, recalling that number of intervals is bounded by constant times 3^m we get for every $\theta \in E$

$$\int_{SSV(\theta,\ell)} |P^\sharp_{2,\theta}(y)|^2 dy \le \frac{C_7\, K}{3^{n-m}}\,.$$

Using the fact that $|2W_m| \le C_8 e^{-40m}$ we estimate

$$I_W = o(\frac{1}{3^{n-m}})\,,\ \text{so}\ I_W = o(\int_{[3^{-m},1]} |P_{2,\theta}(y)|^2\, dy)\,.$$

Recall

Lemma 4. *If $\theta \in J^t$ then $SSV(\theta,\ell) \subset \cup_{i=1}^{c\,3^m} j(i,t,\theta)$.*

Let us now estimate $I_t = S^2 \int_{J^t \cap E} \int_{SSV(\theta,\ell)} \dots$. When we fix t we use Theorem 18. Then either on J^t as a whole, or on one of the subdivision intervals $J^t_u, u = 1, ..., U \le B_0\, m$ we have for each fixed $k = 1, ...K(t)$

$$|g'_1(k,t,\theta| \ge a_2\, \delta_0 \tag{2.8}$$

or the same happens with $g_2(k,t,\theta)$ on the whole J^t_u.

Then again exactly as before, by using $\theta \in E$, we get

$$\int_{SSV(\theta,\ell)} |P^\sharp_{2,\theta}(y)|^2 dy \le \frac{C_7\, K}{3^{n-m}}\,.$$

Now,

$$\int_{J^t_u \cap E} \int_{j(s,k,t,\theta)} |P^\flat_{2,\theta}(y)|^2 |P^\sharp_{2,\theta}(y)|^2\, dy\, d\theta \le$$

$$\int_{J^t_u \cap E} R(y_s(k,t,\theta)(c_2(\theta) - c_1(\theta)))| \int_{j(s,k,t,\theta)} P^\sharp_{2,\theta}(y)|^2\, dy\, d\theta\,,\ s = \kappa, ..., m\,.$$

Here we are using Lemmas 3 and 24.

And now we need to estimate only

$$\int_{J_u^t} R(3^{-s} g_1(k,t,\theta))\, d\theta\,.$$

Notice that we we throw away $\cap E$ at this stage.

We change the variable $v = 3^{-s} g_1(k,t,\theta), \theta \in J_u^t$, and notice that this is a monotone change of variable and (see Theorem 18)

$$\left|\frac{\partial v}{\partial \theta}(\theta)\right| \ge a_2 \delta_0 \cdot 3^{-m}\,.$$

Then a Riesz product observation shows

$$\int_{J_u^t} R(3^{-s} g_1(k,t,\theta))\, d\theta \le (a_2\delta_0 \cdot 3^{-m})^{-1} \int_0^{2\pi} \prod_{k=m+1}^{m+\ell} \frac{7 + 2\cos(3^k\, v)}{9} dv \le C\, 3^m \left(\frac{7}{9}\right)^{\ell}.$$

We already proved ($\theta \in E \cap J_u^t$)

$$\int_{SSV(\theta,\ell)} |P^{\sharp}_{2,\theta}(y)|^2 dy \le \frac{C_7\, K}{3^{n-m}}\,.$$

Therefore,

$$I_t \le 3^m\, S^2\, U \left(\frac{7}{9}\right)^{\ell} \frac{C_7\, K}{3^{n-m}}\,, \tag{2.9}$$

Gathering the estimate $U \le B_0\, m$, the estimate for I_W, and the estimate (2.9) together we obtain by choosing $S = K$ and recalling that $K = 3^m/R$, where R is a large absolute constant:

$$I := \frac{1}{|E|} \int_E \int_{[0,1]\cap SSV(\theta,\ell)} |P^{\flat}_{2,\theta}(y)|^2 |P^{\sharp}_{2,\theta}(y)|^2\, dy\, d\theta \le$$

$$3^m\, S^2 (K 3^{-40m} + B_0\, 3^{B\,m}\, m \left(\frac{7}{9}\right)^{\ell}) \frac{1}{3^{n-m}}\,. \tag{2.10}$$

If we choose

$$\ell = 100\, B\, m\,,$$

we get from (2.10) that for every θ

$$I = o(\frac{1}{3^{n-m}})\,,\; I = o(\int_{[3^{-m},1]} |P^{\flat}_{2,\theta}(y)|^2 |P^{\sharp}_{2,\theta}(y)|^2\, dy\,. \tag{2.11}$$

Therefore

$$\frac{1}{|E|} \int_E \int_{[0,1]\setminus SSV(\theta,\ell)} |P_{2,\theta}(y)|^2\, dy\, d\theta \ge 0.5\, \frac{1}{|E|} \int_E \int_{[3^{-m},1]} |P_{2,\theta}(y)|^2\, dy\, d\theta \ge a\, \frac{1}{3^{n-m}}\,.$$

On the other hand on $[0,1]\setminus SSV(\theta,\ell)$ we have $|P_{1,\theta}(y)|\ge 3^{-A\ell m}=3^{-100AB\,m^2}$ Now we can write

$$C\frac{Km}{N}\ge 3^n\int_{3^{-m}}^1 |P_{1,\theta}(y)|^2|P_{2,\theta}(y)|^2dy\ge C'3^n\,3^{-100AB\,m}\,3^{m-n}\ge C'\frac{3^m}{3^{100AB\,m}},$$

i.e.,

$$m\ge \log\,K\ge C''\frac{N}{3^{100AB\,m}}$$

which implies the contradiction if we choose $m=\epsilon_0\log N$ with sufficiently small ϵ_0, for example, $\epsilon_0=\frac{1}{200AB}$.

Therefore,

$$S=K\le\frac{R}{N^{\epsilon_0}}\,.$$

brings the contradiction to

$$1/|E|=S\,,$$

and, hence,

$$|E|=1/S\le\frac{C}{N^{\epsilon_0}}\,. \tag{2.12}$$

Recall that E was the set of singular directions, on which we do not have overlap of K or larger number of triangles of size $\le 3^{-N}$. Any other direction θ will be in the set of good directions. Such an overlap will happen and (see Section 3)

$$|\mathcal{L}_{\theta,N\,K^c}|\le\frac{C}{K}\,. \tag{2.13}$$

So we proved

Theorem 5.

$$\int_0^\pi|\mathcal{L}_{\theta,n}|\,d\theta\le\frac{C}{n^{\epsilon_0}}\,.$$

3. Combinatorial part

Theorem 6. *If* $|\{x:\max_{0\le n\le N}f_{n,\theta}(x)\ge K\}|\ge\frac{1}{K^\alpha}$ *then*

$$|\mathcal{L}_{\theta,NK^\alpha}|\le\frac{C}{K}\,.$$

Proof. Let $F=\{x:\max_{0\le n\le N}f_{n,\theta}(x)\ge K\}$. We denote by N_x the line orthogonal to direction θ and passing through x. We can call it needle at x. For every $x\in F$ ther are at least K triangles of size $3^{-r}, r=r(x), r\le N$, intersecting N_x. Mark them. Run over all $x \in F$. Consider all marked triangles. Consider all 3^{-N}-triangles that are sub-triangles of marked ones. Call them "green". Let U be a family of green triangles.

We want to show

$$\operatorname{card} U \geq c \cdot K\, |F|\, 3^N\,, \tag{3.1}$$

$$|\operatorname{proj}(\cup_{q\in U} q)| \leq \frac{C}{K} \operatorname{card} U\, 3^{-N}\,, \tag{3.2}$$

Let $\phi := \sum_{q\in U} \chi_q$. Then

$$\int \phi\, dx = \operatorname{card} U\, 3^{-N}\,.$$

Let M denote uncentered maximal function. To prove (3.2) it is enough to show that

$$q \in U \Rightarrow \operatorname{proj} q \subset \{x : M\phi(x) > \frac{K}{C}\}\,,$$

and then to use Hardy–Littlewood maximal theorem. But to prove this claim is easy. In fact, let $x \in \operatorname{proj} q, q \in U$, then there exists Q–the maximal (by inclusion) marked triangle containing q. Consider $I := [x - 10\,\ell(Q), x + 10\,\ell(Q)]$. This segment contains the projections of at least K disjoint triangles $Q_1 := Q, Q_2, ..., Q_K, ...,$ of the same sidelength, which intersect N_{x_0}, where x_0 is a point because of which $Q = Q_1$ was marked. (The readed should see that x_0 lies really well inside I.) So I contains the projections of at least $\frac{\ell(Q)}{\ell(q)} \cdot K$ green traingles. Whence,

$$\int_I \phi\, dx \geq \ell(q) \cdot \frac{\ell(Q)}{\ell(q)} \cdot K \geq \frac{1}{20} |I|\, K\,.$$

So

$$M\phi(x) > \frac{1}{20}\, K\,.$$

We proved (3.2).

Also we proved that $F \subset \{x : M\phi(x) \geq \frac{K}{20}\}$. Therefore, by Hardy–Littlewood maximal theorem

$$|F| \leq |\{x : M\phi(x) \geq \frac{K}{20}\}| \leq \frac{C \int \phi}{K} = C \operatorname{card} U\, 3^{-N}\, K^{-1}\,.$$

This is (3.1).

Let us estimate $|\mathcal{L}_{\theta, N\, K^\alpha}|$ using (3.1) and (3.2). The first step:

$$|\mathcal{L}_{\theta,N}| \leq |\operatorname{proj}(\cup_{q\in U} q)| + 3^{-N}(3^N - \operatorname{card} U) \leq$$
$$\frac{C}{K} \operatorname{card} U\, 3^{-N} + (3^N - \operatorname{card} U) 3^{-N}\,.$$

We do not touch the first term, but we improve the second term by using self-similar structure and going to step $2N$ (inside traingles which are not green there are "green" triangles of size 3^{-2N}). They are just self-similar copies of original green triangles. Then we have the second step:

$$|\mathcal{L}_{\theta,N}| \le \frac{C}{K}\operatorname{card} U\, 3^{-N} + \text{the rest} \le$$

$$\frac{C}{K}\operatorname{card} U\, 3^{-N} + (3^N - \operatorname{card} U)\frac{C}{K}\operatorname{card} U\, 3^{-2N} + (3^N - \operatorname{card} U)^2\, 3^{-2N}\,.$$

Now we leave first two terms alone and having $(3^N - \operatorname{card} U)^2$ traingles of size 3^{-2N} we find again "green" triangles inside each of those, now green traingles of size 3^{-3N}. They are just self-similar copies of original green triangles.

Then we have the third step:

$$|\mathcal{L}_{\theta,3N}| \le \frac{C}{K}\operatorname{card} U\, 3^{-N} + (3^N - \operatorname{card} U)\frac{C}{K}\operatorname{card} U\, 3^{-2N} + \text{the rest} \le$$

$$\frac{C}{K}\operatorname{card} U\, 3^{-N} + (3^N - \operatorname{card} U)\frac{C}{K}\operatorname{card} U\, 3^{-2N} + (3^N - \operatorname{card} U)^2\frac{C}{K}\operatorname{card} U\, 3^{-2N} +$$

$$(3^N - \operatorname{card} U)^3\, 3^{-3N}\,.$$

After the l-th step:

$$|\mathcal{L}_{\theta,l\,N}| \le \frac{C}{K}\operatorname{card} U\, 3^{-N}(1 + (3^N - \operatorname{card} U)3^{-N} + \ldots$$

$$+(3^N - \operatorname{card} U)^{l-1}3^{-(l-1)N}) + (3^N - \operatorname{card} U)^l 3^{-(lN)}\,.$$

So

$$|\mathcal{L}_{\theta,l\,N}| \le \frac{C}{K}\operatorname{card} U\, 3^{-N}\frac{(1-(1-\frac{\operatorname{card} U}{3^N})^l)}{(1-(1-\frac{\operatorname{card} U}{3^N})} +$$

$$e^{-\frac{\operatorname{card} U}{3^N}l} =: I + II\,.$$

Notice that by (3.1) $II \le e^{-K|F|l} \le e^{-K}$ if the step l is chosen to be $l = 1/|F| \le K^\alpha$. However, we always have $I \le \frac{C}{K}$. So Theorem 6 is completely proved.

□

4. The complex analytic part

4.1. **Zeros of $\varphi_\theta(z)$.** In this section (up to factor 3 from before) $\varphi_\theta(z) := e^{-ic_1 z} + e^{-ic_2 z} + e^{-ic_3 z}$, where $c_1 = \cos(\theta - \frac{\pi}{2})$, $c_2 = \cos(\theta - \frac{7\pi}{6})$, $c_3 = \cos(\theta + \frac{\pi}{6})$. We need to know how the zeros are separated and how they behave with changing of $\theta \in [0, 2\pi)$.

Notice that there are three sectors S_1, S_2, S_3 such that, say, $c_1 \geq a$ (a is an absolute positive constant) in S_1 and $c_2, c_3 < 0$ in S_1 (and similarly for other sectors). Sectors have apperture $\pi/3$ each, and are symmetric with respect to rays $\pi/2$, $7\pi/6$, and $-\pi/6$ correspondingly. If, say, $e^{i\theta} \in S_1$ we get that for $z = x + iy$ with $y \geq H = H(a)$, $|\varphi_\theta(z)| \geq 1$. The same for other sectors, so always if $\theta \in S_1 \cup S_2 \cup S_3$ we have

$$|\varphi_\theta(x + iH)| \geq 1\,. \tag{4.1}$$

If we happen to be in $e^{i\theta} \in \bar{S}_1$ then $c_2, c_3 \geq 0$, and $c_1 < -a$. Then,

$$|\varphi_\theta(x - iH)| \geq 1\,. \tag{4.2}$$

Similarly, we could have reasoned that $\varphi_\theta(-z) = \varphi_{\theta+\pi}(z)$. Note also that $|\varphi| \leq C(H)$ when $im(z) \leq H$ (where $C(H)$ is a constant depending on H).

Every rectangle $B := [x_0 - 1, x_0 + 1] \times [-H, H]$ will be called a box. Because of Lemmas 27, 28, and 29, we may say the following:

In every box we have at most absolute constant M of zeros of $\varphi_\theta(z)$ uniformly in $\theta \in [0, 2\pi)$.

For a certain uniform in θ absolute constant $\eta > 0$ we have

$$\{z : \Im z \in (-H, H) : |\varphi_\theta(z)| < \delta\} \subset \cup_{\lambda_i} D(\lambda_i, \delta^\eta)\,. \tag{4.3}$$

Here $\{\lambda_i = \lambda_i(\theta)\}$ are zeros of φ_θ.

Notice also that uniformly in θ for all sufficiently large m

$$|\{i : |\lambda_i| \leq 3^m|\} \leq C\, 3^m\,. \tag{4.4}$$

The constant C is absolute and uniform in θ. (This last fact could have also been obtained from the theory of entire functions with growth conditions)

4.2. **Zeros of $\varphi_\zeta(z)$.** Now $\zeta = \theta + i\sigma$, $\varphi_\zeta(z) = e^{-ic_1(\zeta)z} + e^{-ic_2(\zeta)z} + e^{-ic_3(\zeta)z}$.

Recall that $c_1(\zeta) = \cos(\zeta - \frac{\pi}{2})$, $c_2 = \cos(\zeta - \frac{7\pi}{6})$, $c_3 = \cos(\zeta + \frac{\pi}{6})$, $c_1 = \cos(\theta - \frac{\pi}{2})$, $c_2 = \cos(\theta - \frac{7\pi}{6})$, $c_3 = \cos(\theta + \frac{\pi}{6})$,and $s_1 = \sin(\theta - \frac{\pi}{2})$, $s_2 = \sin(\theta - \frac{7\pi}{6})$, $s_3 = \sin(\theta + \frac{\pi}{6})$. Then we write ($\zeta = \theta + i\sigma$)

$$\varphi_\zeta(z) = e^{-ic_1 \cosh\sigma\,(x+iy)} e^{-s_1 \sinh\sigma\,(x+iy)} + e^{-ic_2 \cosh\sigma\,(x+iy)} e^{-s_2 \sinh\sigma\,(x+iy)} +$$

$$e^{-ic_3 \cosh\sigma\,(x+iy)} e^{-s_3 \sinh\sigma\,(x+iy)}\,.$$

Fix

$$Q = \{\zeta : -\pi/2 < \theta < 5\pi/2, -\sigma_0 < \sigma < \sigma_0\}\,.$$

where σ_0 is a small positive absolute constant. By box we will understand

$$B = B_{x_0} := \{z : x_0 - 1 < x < x_0 + 1, -H < y < H\}\,.$$

Lemma 7. *Thre is an absolute constant M such that for any $\zeta \in Q$ and for any box $B_{x_0}, x_0 > 0$,*

$$\mathit{card}(\lambda \in B : \varphi_\zeta(\lambda) = 0) \le M\,. \tag{4.5}$$

Proof. This is again Lemma 27. We want $|\varphi|$ to have an upper bound on the boundary, and a lower bound at a point.

Consider first the case of $\sigma \ge 0$. In a box B_{x_0} we have the estimate from above

$$|\varphi_\zeta(z)| \le C_0\,,$$

if $x_0 \sinh\sigma \le 10$. If $x_0 \sinh\sigma \ge 10$ we have

$$|\varphi_\zeta(z)| \le C_0\, e^{s_i \sinh\sigma\, x_0}$$

for some $i = 1, 2, 3$. We want to prove that the box contains a point w such that

$$|\varphi_\zeta(w)| \ge c_0\,,$$

if $x_0 \sinh\sigma \le 10$. If $x_0 \sinh\sigma \ge 10$ we have

$$|\varphi_\zeta(w)| \le c_0\, e^{s_i \sinh\sigma\, x_0}$$

for the same $i = 1, 2, 3$. The case $x_0 \sinh\sigma \le 10$ is treated exactly as in (4.1), (4.2), and as a result, we find $w = x_0 \pm H$ satisfying $|\varphi_\zeta(w)| \ge c_1$. May be only H should be chosen bigger, but its size is not dependent on anything (except number 10).

If $x_0 \sinh\sigma \ge 10$, consider several cases. Real line is split to intervals of length π, we call such interval positive if sin on it is positive. Notice that given a positive interval only one or two of $\theta - \pi/2, \theta - 7\pi/6, \theta + \pi/6$ belongs to it $\mathit{mod}\, 2\pi$.

Case 1 is when only one, say, $\theta - \pi/2$ belongs to a positive interval $mod\, 2\pi$. Notice that then $s_1 \geq \sqrt{3}/2$, $s_2, s_3 \leq 0$, $\sinh\sigma\, x_0 \geq 10$, and as a result

$$|\varphi_\zeta(x_0)| \geq e^{s_1 \sinh\sigma\, x_0} - 2 \geq \frac{1}{2} e^{s_1 \sinh\sigma\, x_0}\,.$$

Case 2 is when both elements of a pair, say s_1, s_2, are non-negative, $s_3 \leq 0$. Consider the situation when $s_1 > s_2 + 1/10$. Then it is easy to see that $s_1 \geq 1/2$, then

$$|\varphi_\zeta(x_0)| \geq e^{s_1 \sinh\sigma\, x_0}(1 - e^{-1/10 \sinh\sigma\, x_0}) - 1 \geq (1-1/e)e^{s_1 \sinh\sigma\, x_0} - e^{-5} e^{s_1 \sinh\sigma\, x_0} \geq c_0 e^{s_1 \sinh\sigma\, x_0}\,.$$

We are left to consider the situation when s_1, s_2 are non-negative, $s_3 \leq 0$, and $s_2 \leq s_1 \leq s_2 + 1/10$. Notice that in this case $|c_1|, |c_2| \geq 1/3$. Suppose $c_1 > 0$, choose $h > 0$ such that $e^{c_1 h} \geq 2$. It is enough to choose $h = 3$. Consider $w = x_0 + ih$. If there would be $c_1 < 0$ (and so $c_1 < -1/3$) we would choose $w = x_0 - ih$. In both cases, c_2 has an opposite sign, and we can notice that $s_3 \leq -\sqrt{3}/2$. Hence,

$$|\varphi_\zeta(x)| \geq 2\, e^{s_1 \sinh\sigma\, x_0} - e^{s_2 \sinh\sigma\, x_0} - e^{|c_3| \cosh\sigma_0\, h} e^{-\frac{\sqrt{3}}{2} \sinh\sigma\, x_0}\,.$$

Using the facts that $\sigma_0 \leq 1/10$, $\sinh\sigma\, x_0 \geq 10$, $h = 3$, $s_2 \leq s_1$, $s_1 \geq 1/2$, we get from the above:

$$|\varphi_\zeta(x)| \geq e^{s_1 \sinh\sigma\, x_0} - e^{1/30} e^{-\frac{\sqrt{3}}{2} \sinh\sigma\, x_0} \geq \frac{1}{2}\, e^{s_1 \sinh\sigma\, x_0}\,.$$

We finished to consider the case of $\sigma \geq 0$, the case $\sigma < 0$ is taken care of in a symmetric fashion.

To finish the estimate (4.5) we notice that In a doubled box $2\, B_{x_0}$ we have the estimate from above

$$|\varphi_\zeta(z)| \leq C_0\,, \tag{4.6}$$

if $x_0 \sinh\sigma \leq 10$. If $x_0 \sinh\sigma \geq 10$ we have

$$|\varphi_\zeta(z)| \leq C_0\, e^{s_i \sinh\sigma\, x_0} \tag{4.7}$$

for some $i = 1, 2, 3$. We have already proved that the box B_{x_0} with sufficiently large absolute $H \geq 3$ contains a point w such that

$$|\varphi_\zeta(w)| \geq c_0\,, \tag{4.8}$$

if $x_0 \sinh\sigma \leq 10$. If $x_0 \sinh\sigma \geq 10$ we proved the existence of $w \in B_{x_0}$ such that

$$|\varphi_\zeta(w)| \leq c_0\, e^{s_i \sinh\sigma\, x_0} \tag{4.9}$$

for the same $i = 1, 2, 3$. Here c_0, C_0 are absolute constants.

Lemma 27 now applies to all of our cases. □

4.3. **The set of small values of $P_{1,\theta}(y)$.** In this section we want to investigate the set

$$\mathcal{G} := \{y \in [3^{-m}, 1] : |P_{1,\theta}(y)(y)| = |\varphi_\theta(y)\varphi_\theta(3y) \cdot ... \cdot \varphi_\theta(3^m y)| \geq 3^{-A\,m\ell}\}\,.$$

If $\Omega(k, \theta, 3^{-A\ell}) := \{y \in [3^{-m}, 1] : |\varphi_\theta(3^k y)| < 3^{-A\ell}$, then the set of small values

$$\Omega(\theta, \ell) = [3^{-m}, 1] \setminus \mathcal{G} \subset \cup_{k=0}^m \Omega(k, \theta, 3^{-A\ell})\,.$$

We already saw that if $A \geq 1/\eta$

$$\{z = x + iy, 0 < x < 3^m, -H < y < H : |\varphi_\theta(z)| < 3^{-A\ell} \subset \cup_i D(\lambda_i(\theta), 3^{-\ell})\}\,,$$

where $\lambda_i(\theta)$ are zeros of $\varphi_\theta(z)$. In proving this we essentially used only the absolute bound on the number od zeros of φ_θ in the box, and the fact that each box has a point where $|\varphi_\theta(w)|$ is comparable with the max $|\varphi_\theta|$ over the box.

But the same is formulated for $\varphi_\zeta(z)$, $\zeta \in Q$, in Lemma 7 and in (4.6), (4.8), (4.7), (4.9), so we trapped the set of small values of φ_ζ in the collection of discs:

$$\{z = x + iy, 0 < x < 3^m, -H < y < H : |\varphi_\zeta(z)| < 3^{-A\ell} \subset \cup_i D(\lambda_i(\zeta), 3^{-\ell})\}\,,$$

where $\lambda_i(\zeta)$ are zeros of $\varphi_\zeta(z)$.

Now we want to show that $P_{2,\theta}(y)$ is still large enough away from the set where $P_{1,\theta}(y)$ is small. Let $\Omega(\theta, \ell) := \{y \in [3^{-m}, 1] : |P_{1,\theta}(y)| \leq C 3^{-A\,\ell\,m}\}$. Then $\Omega(\theta, \ell)$ is contained in contractions by the factors 3^{-k}, $k = 1, 2, ..., m$, of $3^{-\ell}$-neighborhood of the complex zeroes of $\phi_\theta(z)$. By (4.4) and by this whole subsection $\Omega(\theta, \ell) \subset \bigcup_{j=1}^L J_j(\theta)$, where $L \leq C\, 3^m$ and $|J_j(\theta)| \leq C\, 3^{-k-\ell}$ $(k = k(j) \in \{1, 2, ..., m\})$.

4.4. **Branch points of ϕ_θ.** We call a point $\zeta = \theta + i\sigma \in Q$ a *branch point* of $\phi_\zeta(z)$ if there exists $z \in [-3^m - 1, 3^m + 1] \times [-H/2, H/2]$ such that

$$\begin{cases} \phi_\zeta(z) = 0 \\ \frac{\partial}{\partial z}\phi_\zeta(z) = 0\,. \end{cases} \tag{4.10}$$

Lemma 8. *For real $\zeta = \theta \in Q \cap \mathbb{R}$ there are no branch points.*

Proof. We will prove more: that for $\zeta = \theta \in Q \cap \mathbb{R}$ the system (4.10) has no solutions z at all.

As always $c_1 = \cos(\theta - \pi/2), c_2 = \cos(\theta - 7\pi/6), c_3 = \cos(\theta + \pi/6)$, also

$$b := (c_2 - c_1)/\sqrt{3} = \sin(\theta - 5\pi/6), a := (c_1 - c_3)/\sqrt{3} = \sin(\theta - \pi/6), -(a+b) = (c_3 - c_2)/\sqrt{3} = \cos\theta\,.$$

If (4.10) is valid then

$$\begin{cases} e^{iZb} + e^{-iZa} = -1 \\ be^{iZb} - ae^{-iZa} = 0\,. \end{cases} \tag{4.11}$$

Hence

$$\begin{cases} e^{iZb} = -\frac{a}{a+b} \\ e^{-iZa} = -\frac{b}{a+b} \end{cases} \tag{4.12}$$

has a solution $(Z = \sqrt{3}z)$. Here $a, b \in \mathbb{R}$. If $a = 0$ or $b = 0$ or $a + b = 0$ there is no solution of (4.12) just because the exponent cannot be zero or infinity.

Suppose all these three numbers do not vanish. Take absolute values in (4.5):

$$\begin{cases} e^{-bY} = \frac{|a|}{|a+b|} \\ e^{aY} = \frac{|b|}{|a+b|} \\ e^{-(a+b)Y} = \frac{|a|}{|b|}\,. \end{cases} \tag{4.13}$$

Consider the cases:

1. $a > 0, b > 0$. Then the first gives $bY > 0$, and $b > 0$, so $Y > 0$. The second gives $aY < 0$, and $a > 0$, so $Y < 0$. Contradiction.

2. $a < 0, b < 0$. Then the first gives $bY > 0$, and $b < 0$, so $Y < 0$. The second gives $aY < 0$, and $a < 0$, so $Y > 0$. Contradiction.

3. $a > 0, b < 0, a + b > 0$. Then $|a| > |b|$. Then the first gives $-bY > 0$, and $b < 0$, so $Y > 0$. The third gives $-(a + b)Y > 0$, and $a + b > 0$, so $Y < 0$. Contradiction.

4. $a > 0, b < 0, a + b < 0$. Then $|a| < |b|$. Then the second gives $aY > 0$, and $a < 0$, so $Y < 0$. The third gives $-(a + b)Y < 0$, and $a + b < 0$, so $Y < 0$. Contradiction.

5. $a < 0, b > 0, a + b > 0$. Then $|b| > |a|$. Then the second gives $aY > 0$, and $b < 0$, so $Y > 0$. The third gives $-(a + b)Y > 0$, and $a + b > 0$, so $Y > 0$. Contradiction.

6. $a < 0, b > 0, a + b < 0$. Then $|a| > |b|$. Then the first gives $-bY > 0$, and $b > 0$, so $Y < 0$. The third gives $-(a + b)Y > 0$, and $a + b < 0$, so $Y > 0$. Contradiction.

□

Actually we just proved a little bit more. To formulate it we need some notations. Let $W := \{\pi/2, 5\pi/6, \pi/6 \ (mod\, \pi)\} \cap Q$. It is a finite set, let $W_m := \cup_{w\in W} D(w, 2 \cdot 3^{-40m})$.

Recall the above definition of $a(\theta), b(\theta)$ and put ($Z = \sqrt{3}z$, $Z = X + iY$)

$$d_\theta(z) := \max\left(\left||e^{iZb(\theta)} \cdot \frac{|a(\theta) + b(\theta)|}{|a(\theta)|} - 1\right|, \left||e^{-iZa(\theta)} \cdot \frac{|a(\theta) + b(\theta)|}{|b(\theta)|} - 1\right|\right).$$

$$d_\theta := \inf_{z\in\mathbb{C}} d_\theta(z)\,.$$

This is called discrepancy. We actually proved the following estimate for the discrepancy.

Lemma 9. *There is an absolute positive constant c such that $\min_{\theta\in Q\setminus W_m} d_\theta \geq c\,3^{-40m}$.*

Similarly for complex $\zeta = \theta + i\sigma$ we have

$$b := \sin(\theta + i\sigma - 5\pi/6), a := \sin(\theta + i\sigma - \pi/6)\,.$$

$$\Im b = \cos(\theta - 5\pi/6) \cdot \sinh\sigma\,,\ \Im a = \cos(\theta - \pi/6) \cdot \sinh\sigma\,. \tag{4.14}$$

We introduce

$$d_\zeta(z) := \max\left(\left||e^{iZb(\zeta)} \cdot \frac{|a(\zeta) + b(\zeta)|}{|a(\zeta)|} - 1\right|, \left||e^{-iZa(\zeta)} \cdot \frac{|a(\zeta) + b(\zeta)|}{|b(\zeta)|} - 1\right|\right).$$

$$d_\zeta := \inf_{z\in[-3^m-1,3^m+1]\times[-H/2,H/2]} d_\zeta(z)\,.$$

4.5. **Branch points of ϕ_ζ.** If we leave the real axis and venture $\zeta = \theta + i\sigma$ into a complex domain we acquaire the factor $e^{\pm\cos(\theta-5\pi/6)\,X\,\sinh\sigma}$ into $|e^{iZb}|$ and the factor $e^{\pm\cos(\theta-\pi/6)\,X\,\sinh\sigma}$ into $|e^{-iZa}|$. This is from (4.14). Clearly it is very close to 1 if $|X| \leq 3^m + 1$ and $|\sigma| \leq 3^{-100m}$. The change ratios $|a(\theta)|/|a(\theta+i\sigma)|, |b(\theta)|/|b(\theta+i\sigma)|$ will also be very close to 1 if $\theta, \theta+i\sigma \in Q\setminus W_m$, and $|\sigma| \leq 3^{-100m}$. They differ from 1 by at most $C\,3^{-96m}$. Therefore we proved

Lemma 10. *Let $\zeta \in Q \setminus W_m, |\Im\zeta| \leq 3^{-100m}$. Then $d_\zeta \geq \frac{c}{2}3^{-40m}$, where c is the absolute constant from Lemma 9.*

4.6. **Zeros of ϕ_ζ as analytic functions:** $\Lambda : \zeta \to \Lambda(\zeta)$**.** Let us fix a point $\theta_0 \in Q \setminus 2\,W_m$, and consider the disc $D(\theta_0) := D(\theta_0, 2\cdot 3^{-Bm})$, where $B \geq 100$ and will be chosen later. For any $\zeta \in \bar{D}(\theta_0, 2\cdot 3^{-Bm})$ (the closure of the disc) we consider zeros $Z_{m,H/4}(\zeta) := \{\lambda_i(\zeta), i = 1, ..., I(\zeta)\}$ of ϕ_ζ lying in $[-3^m, 3^m] \times [-H/4, H/4]$. We know that there exists an absolute constant M independent of θ_0, ζ such that

$$\mathrm{card}(Z_{m,H} \cap [x-1, x+1] \times [-3/2H, 3/2H]) \leq M\,,\ forall x \in [-3^m, 3^m]\,. \quad (4.15)$$

Lemma 11. *These are continuous functions on* $\bar{D}(\theta_0, 2\cdot 3^{-Bm})$*.*

Proof. Let $\zeta \in \bar{D}(\theta_0, 2\cdot 3^{-Bm})$. All points in $Z_{m,H}(\zeta)$ are simple zeros, this follows from Lemma 10, for example. Let $\eta(\zeta) := \min_{i\neq j, i,j \leq I(\zeta)} |\lambda_i(\zeta) - \lambda_j(\zeta)|$. Then $\eta > 0$. Fix i, call $\lambda = \lambda_i(\zeta)$. We know that there exists an absolute constant M independent of θ_0, ζ such that

$$\mathrm{card}(Z_{m,H}(\zeta) \cap [\lambda - 1, \lambda + 1] \times [-H, H]) \leq M\,. \quad (4.16)$$

Consider the discs of radius $\eta/2, \eta \in (0, \eta(\zeta))$ around λ and around other points in $Z_{m,H}(\zeta) \cap [\lambda-1, \lambda+1] \times [-H, H]$. Call them $B_0, ..., B_{M'}$, $M' \leq M$, B_0 is the one centered at λ. We also know that

$$|\phi_\zeta(z)| \geq a\,(\eta/2)^M\,,\ \forall z \in [\lambda-1, \lambda+1] \times [-H, H] \setminus \cup_{s=1}^{M'} B_s\,. \quad (4.17)$$

Obviously

$$|\phi_\zeta(z) - \phi_{\zeta'}(z)| \leq C_0\,|\zeta - \zeta'| \cdot 3^m\,, \quad (4.18)$$

$$\forall \zeta, \zeta' \in \bar{D}(\theta_0), z \in [-3^m - 1, 3^m + 1] \times [-3/2H, 3/2H]\,. \quad (4.19)$$

Hence, if ζ' is very close to ζ, namely

$$|\zeta' - \zeta| \leq \frac{a}{3C_0}\,(\eta/2)^M \cdot 3^{-m}$$

we get that

$$|\phi_\zeta(z)| > |\phi_\zeta(z) - \phi_{\zeta'}(z)|\,,\ \forall z \in \cup_{s=1}^{M'} \partial B_s\,.$$

So these functions: $\phi_\zeta(z), \phi_{\zeta'}(z)$ have the same number of zeros in each B_s. We need $s = 1$, B_1 being centered at $\lambda = \lambda_i(\zeta)$. We conclude that

$$\zeta' \in B(\lambda, \eta(\zeta)/2) \to \lambda(\zeta')$$

is a continuous function of ζ' at ζ:

$$|\lambda(\zeta') - \lambda(\zeta)| \leq (3^{m+1} C_0 |\zeta' - \zeta|/a)^{1/M}\,. \quad (4.20)$$

Lemma is proved.

□

Definition.

$$\delta_1 := \min\{|\lambda_i(\zeta) - \lambda_j(\zeta)| : i \neq j,\, i, j \leq I(\zeta),\, \zeta \in \bar{D}(\theta_0)\,.$$

We proved

$$\delta_1 > 0\,. \tag{4.21}$$

Given a radial path $p = [\theta_0, \theta_0 + re^{it}] = [\theta_0, \zeta], r < 2\cdot 3^{-Bm}$ and one $i \leq I(\theta_0)$ we can now consider a well-defined and continous function $t \in [0, r] \to \lambda_i(\theta_0 + te^{it})$. So we extend $\lambda_i(\theta_0)$ to a function $\lambda_i(\zeta), \zeta \in D(\theta_0, 2\cdot 3^{-B'm})$, $i = 1, ..., I(\theta_0)$. These are all single valued analytic function in this disc. In fact, let us see that a just defined $\lambda_i(\zeta)$ satisfies

$$|\lambda_i(\zeta) - \lambda_i(\theta_0)| \leq C_1\, 3^{-B'm}\,,\ \forall \zeta \in D(\theta_0, 2\cdot 3^{-Bm})\,. \tag{4.22}$$

Suppose (4.22) is already proved. We saw how to extend the analytic germ of $\lambda_i(\cdot)$, by monodromy theorem we would have a single valued analytic function in $D(\theta_0, 2\cdot 3^{-Bm})$ if we can show that we do not meet branch points while extending to $\zeta \in D(\theta_0, 2\cdot 3^{-Bm})$ along the paths inside $D(\theta_0, 2\cdot 3^{-Bm})$. But if we choose B' large enough, then (4.22) shows that the extension $\lambda_i(\zeta)$ is still in $[-3^m - 1, 3^m + 1] \times [-H, H]$. So by Lemma 10 we could not meet branch points.

We are left to prove (4.22). We use Rouché's theorem again. We fix $i \leq I(\theta_0)$ and denote $\lambda := \lambda_i(\theta_0)$ as before in Lemma 11.

Consider the discs $D_s(\lambda(\theta_0), 3^{-B'm})$, $s = 1, ..., M'$ around zeros of $\phi_{\theta_0}(z)$ lying in $[\lambda - 1, \lambda + 1] \times [-H, H]$. Unlike Lemma 11 they may be not disjoint. But the number of them–M' is still at most M, where M is an absolute constant. Let Ω be a connected component of $\cup_{s=1}^{M'} D_s(\lambda(\theta_0), 3^{-B'm})$ containing λ. Obviously

$$\operatorname{diam}\Omega \leq M\, 3^{-B'm}\,. \tag{4.23}$$

Let $\zeta = \theta_0 + re^{it}$, let γ is a continuous path $\lambda_i(\theta_0 + ue^{it}), u \in [0, r], r < 2\cdot 3^{-Bm}$. It starts in Ω, but suppose it hits the boundary of Ω for $t = t_0$. Denote $\zeta_0 = \theta_0 + t_0 e^{it}$.

$$|\phi_{\theta_0}(z)| \geq a\, 3^{-B'Mm}\,,\ \forall z \in [\lambda - 1, \lambda + 1] \times [-H, H] \setminus \cup_{s=1}^{M'} D_s\,. \tag{4.24}$$

Obviously

$$|\phi_{\theta_0}(z) - \phi_{\zeta_0}(z)| \leq C_0\, |\theta_0 - \zeta_0| \cdot 3^m \leq 2C_0\, 3^m\, 3^{-Bm}\ , \tag{4.25}$$

$$\forall \zeta_0 \in \bar{D}(\theta_0, 2\cdot 3^{-Bm}),\ z \in [-3^m - 1, 3^m + 1] \times [-3/2H, 3/2H]\,. \tag{4.26}$$

Notice that

$$\lambda_i(\theta_0 + t_0 e^{it}) \in [-3^m - 1, 3^m + 1] \times [-3/2H, 3/2H]$$

because of (4.23). Then if $B > 10\,B'M$, the fact that $\phi_{\zeta_0}(\lambda_i(\theta_0 + t_0 e^{it})) = 0$ contradicts the combination of (4.24) and (4.25) at $z := \lambda_i(\theta_0 + t_0 e^{it}) \in \partial\Omega \subset \cup_{s=1}^{M'} \partial D_s$.

So our continuous path never hits $\partial\Omega$. Hence for $\zeta = \theta_0 + re^{it}, r < 2 \cdot 3^{-Bm}$, the point $\lambda_i(\zeta) \in \Omega$. Then (4.23) proves (4.22).

4.7. **Estimates of analytic functions $\zeta \in D(\theta_0, 3^{-Bm}) \to \lambda(\zeta)$.** We choose the constant $\delta_0 > 0$ is such that 1) in $B(0, \delta_0)$ there are no zeros of any function ϕ_θ, 2) $\delta_0 < H/10$.

We again fix $\theta_0 \in Q \cap \mathbb{R} \setminus 2W_m$, consider the discs $D(\theta_0) := D(\theta_0, 2 \cdot 3^{-Bm})$ and $D'(\theta_0) := D(\theta_0, 1.5 \cdot 3^{-Bm})$. Consider zeros of $\phi_\zeta(z), \zeta \in D(\theta_0), z \in [-3^m, 3^m] \times [-\delta_0/10, \delta_0/10]$, call them $\{\lambda_i(\zeta)\}_{i=1}^{I'(\theta_0)}$, and notice that if B is sufficiently large, then

$$|\lambda_i(\zeta) - \lambda_i(\theta_0)| \le 3^{-\frac{1}{20M} Bm}, \ \forall \zeta \in D(\theta_0), \, i = 1, .., I'(\theta_0) \,. \tag{4.27}$$

Here M is an absolute bound on a number of zeros used above. This comes from (4.22) by carefully looking at how we chose B' in (4.22).

Recall $c_1(\zeta) = \cos((\zeta) - \frac{\pi}{2})$, $c_2(\zeta) = \cos((\zeta) - \frac{7\pi}{6})$, $c_3(\zeta) = \cos((\zeta) + \frac{\pi}{6})$.
Definition. Fix $i = 1, ..., I'(\theta_0)$, put

$$g_1(\zeta) := g_{1,i}(\zeta) = \frac{1}{2}(\lambda_i(\zeta) + \bar{\lambda}(\bar{\zeta}))(c_2(\zeta) - c_1(\zeta))$$
$$g_2(\zeta) := g_{2,i}(\zeta) = \frac{1}{2}(\lambda_i(\zeta) + \bar{\lambda}(\bar{\zeta}))(c_3(\zeta) - c_1(\zeta)) \,.$$

Lemma 12. *$|g_1(\zeta)|, |g_2(\zeta)|, |g_1'(\zeta)|, |g_2'(\zeta)| \le C_2\, 3^{aBm}$, where a, C_2 are absolute positive and finite, and ζ is any point of $D'(\theta_0)$.*

Proof. This follows from (4.27), the fact that $\lambda_i(\theta_0)| \le 3^m$, and the fact that all g_1, g_2 are analytic functions in $D(\theta_0)$.

□

Let $D''(\theta_0) := D(\theta_0, 3^{-Bm})$.

Lemma 13. *Either*

$$card\{\zeta \in D''(\theta_0) : g_1'(\zeta) = 0|\} \le B_0(B)\, m\,, \tag{4.28}$$

or

$$\|g_1'\|_{L^\infty(D''(\theta_0))} \le 3^{-Bm}\,. \tag{4.29}$$

Proof. We have an analytic function $f = g_1'$ in the disc D'. It is bounded by $L = C_2 3^{aBm}$. Two things may happen: at a certain point of $a \in 2/3D'$ it is begger than 3^{-Bm}. We write the Jensen's inequality for $\log|f|$ in D'. Then the number of zeros in $2/3D'$ will be estimated by $A(\log L - \log(3^{-Bm})$, which is less than $B_0(B)m$ in our case. So lemma's dichotomy is proved.

□

Lemma 14. *For every* $i = 1, ..., I'(\theta_0)$ *and every* $\theta \in Q \cap \mathbb{R}$ *we have*

$$\max(|g_1'(\theta)|, |g_2'(\theta)|) \ge a_1\, \delta_0\,.$$

with a positive absolute a_1.

Proof. We use the notations:

$$Y(\theta) := Y_i(\theta) = \frac{1}{2}(\lambda_i(\theta) + \bar{\lambda}_i(\theta))\,.$$

Also $g_1'(\theta) = Y'(c_2 - c_1) - Y(s_2 - s_1)$, and $g_2'(\theta) = Y'(c_3 - c_1) - Y(s_3 - s_1)$. If $\max(|g_1'(\theta)|, |g_2'(\theta)|) \le \epsilon$, then it follows that

$$|\frac{s_3 - s_1}{c_3 - c_1} - \frac{s_2 - s_1}{c_2 - c_1}| \le \frac{\epsilon(|c_3 - c_1| + |c_2 - c_1|)}{Y|c_3 - c_1||c_2 - c_1|}.$$

We get $Y \le C\epsilon$. Simply one can check that $(c_3-c_1)(c_2-c_1)[\frac{s_3-s_1}{c_3-c_1} - \frac{s_2-s_1}{c_2-c_1}] = 3\sqrt{3}/2$.

This formula works for equilateral triangle. If we have *any* non-degenerate triangle normalized to have the radius of ascribed circle to be 1, we can think that the verices of the triancle are in $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$, and we would easily get

$$|((c_3-c_1)(s_2-s_1)-(c_2-c_1)(s_3-s_1)| = 4|\sin(\frac{\theta_1 - \theta_2}{2}||\sin(\frac{\theta_2 - \theta_3}{2}||\sin(\frac{\theta_3 - \theta_1}{2}| \ne 0\,.$$

The same non-degeneracy of determinant as in the eqilateral case follows. We get the same conclusion We get $Y \le C\epsilon$.

But recall that $Y(\theta) = \Re\lambda_i(\theta), i = 1, ..., I'(\theta)$, so $|\Im\lambda_i(\theta)| \le \delta_0/10$. But $|\lambda_i(\theta)| \ge \delta_0$ by the definition of δ_0. So $|Y(\theta)| = |\Re\lambda_i(\theta)| \ge \frac{9}{10}\delta_0$.

□

Definition. $J(\theta_0) := Q \cap \mathbb{R} \cap D(\theta_0, 3^{-Bm})$.

Lemma 15. *If for a given* $\theta_0 \in Q \cap \mathbb{R} \setminus 2W_m$ *in Lemma 13 we have* (4.29)*, then* $|g_2'(\theta)| \ge a_1\delta_0\,,\ \forall \theta \in J(\theta_0)$.

Proof. Obvious from Lemma 14. □

We proved

Lemma 16. *For a given* $\theta_0 \in Q \cap \mathbb{R} \setminus 2W_m$ *and* $i = 1, ..., I'(\theta_0)$ *we can have either 1)* (4.28) *for* $g_{1,i}'$ *and* $g_{2,i}'$ *simultaneously, or 2)* $|g_{1,i}'(\theta)| \ge a_1\delta_0\,,\ \forall\theta \in J(\theta_0)$*, or 3)* $|g_{2,i}'(\theta)| \ge a_1\delta_0\,,\ \forall\theta \in J(\theta_0)$.

Notice that in all three cases

$$\max_{x\in\mathbb{R}} \operatorname{card}(J_{\theta_0} \cap g_1^{-1}(x)) \le B_0 m\,, \text{ or } \max_{x\in\mathbb{R}} \operatorname{card}(J_{\theta_0} \cap g_2^{-1}(x)) \le B_0 m\,. \tag{4.30}$$

(In the first case both relationships of (4.30) hold simultaneously.) In fact, notice that for a smooth real f the cardinality of the pre-image (also called the multiplicity of covering of image) is bounded by the number of zeros of the derivative. In case 2) g_1' does not have any zeros, the same holds in case 3) for g_2'. In case 1) we have the bound on the number of zeros of both g_1', g_2'.

In a certain sense (4.30) is the main claim for the sake of which we launched into investigation of analytic branches of zeros of $\phi_\zeta(z)$. We will need (4.30) soon, but actually we need a bit more.

Suppose we are *not* in the case 2) or 3). Consider functions $p(\zeta) = g_1(\zeta) + g_2(\zeta), m(\zeta) = g_1(\zeta) - g_2(\zeta)$. Then we have the analog of Lemma 13:

Lemma 17. *Either*

$$card\{\zeta \in D'(\theta_0) : p'(\zeta) = 0\} \le B_0 m\,,\ and\ card\{\zeta \in D'(\theta_0) : m'(\zeta) = 0\} \le B_0 m\,, \tag{4.31}$$

or

$$\min(\|m'\|_{L^\infty(D'(\theta_0))}, \|p'\|_{L^\infty(D'(\theta_0))}) \le 3^{-Bm}\,. \tag{4.32}$$

The proof is the same as for Lemma 13.

If, for example (4.32) happens and, say, $\|m'\|_{L^\infty(D'(\theta_0))} \le 3^{-Bm}$, then

$$||g_1'| - |g_2'|| \le 3^{-Bm}$$

everywhere on $J(\theta_0)$. Combining this with Lemma 14, and choosing large B we conclude that

$$|g_1'| \geq a_1/2\delta_0 \text{ and } |g_1'| \geq a_1/2\delta_0 \text{ everywhere on } J_(\theta_0)\,.$$

So we are back to cases 2) and 3) (simultaneously) of Lemma 16.

On the other hand, if (4.31) happens then the number of points $\theta \in J(\theta_0)$ such that

$$|g_1'(\theta)| = |g_2'(\theta)|$$

is bounded by $2B_0 m$. In fact, this equality for real $g_1'(\theta), g_2'(\theta)$ may happen only if either $g_1'(\theta) = g_2'(\theta)$ (so $m'(\theta = 0)$ or $g_1'(\theta) = -g_2'(\theta)$ (and so $p'(\theta) = 0$). And the number of such points is bounded in (4.31).

In this latter case, we subdivide $J(\theta_0)$ to intervals $J(s, \theta_0), s = 1, ..., B_1 m$ according to whether

$$|g_1'(\theta)| \geq |g_2'(\theta)| \text{ or } |g_2'(\theta)| \geq |g_1'(\theta)|$$

everywhere on interval J_s.

Theorem 18. *For every $\theta_0 \in Q \cap \mathbb{R} \setminus 2W_m$ we can subdivide interval $J(\theta_0) = \mathbb{R} \cap D(\theta_0, 3^{-Bm})$ into at most $B_1 m$ intervals (may be we even do not need to subdivide at all) $J(s, \theta_0)$ such that on each of them at least one of element of each pair $g_{1,i}(\theta), g_{2,i}(\theta)$, $i = 1, ... I'(\theta_0)$, is monotone and the modulus of its derivative is at least $a_2\delta_0$.*

4.8. **The set of small values of $P_{1,\theta}(y)$ revisited.** Of course

$$SSV(\theta, \ell\, m) \subset \cup_{s=1}^m SSV(s, \theta, \ell)\,,\ SSV(s, \theta, \ell) := \{y \in [0,1] : |\phi_\theta(3^s\, y)| < 3^{-A\,\ell}\}\,.$$

To understand $SSV(s, \theta, \ell)$ let us make some notations first.

$$R_k := [3^k, 3^{k+1}] \times [-\delta_0/10, \delta_0/10],\ k = 1, ..., m-1\,,\ R_0 := [0,1] \times [-\delta_0/10, \delta_0/10]\,.$$

$$\omega_k(\theta) := \cup_{\lambda(\theta) \in R_k \text{ is a zero of } \phi_\theta} B(\lambda(\theta), 3^{-\ell})\,.$$

Consider

$$G_k(\theta) := 3^{-k}\omega_k(\theta)\,.$$

Finally, let

$$G(\theta) := \cup_{k=0}^m G_k(\theta)\,.$$

4.9. **Putting the set of small values into a small collection of intervals.**

Lemma 19. $SSV(s,\theta,\ell)\subset\cup_{k=1}^{s}G_k(\theta)$.

Proof. Choose $y\in[0,1]$. Suppose $y\in[0,1]\setminus\cup_{k=1}^{s}G_k$. Let s be a number such that $3^s y\in R_s$. If $y\in 3^{-s}\omega_s$, then $y\in G_s$. We asumed the contrary, so $y\notin 3^{-s}\omega_{k'}$. But then $3^s\,y$ is not in any disc centered at a zero of ϕ_θ in $[y-1,y+1]\times[-\delta_0/10,\delta_0/10]$ and radius $3^{-\ell}$. Other such discs (not counted in $\omega_s(\theta)$) are just far enough from the real axis to contain $3^s\,y\in\mathbb{R}$. Then

$$|\phi_\theta(3^s\,y)|\ge 3^{-A\,\ell}$$

by Lemma 28. So y is not in $SSV(s,\theta,\ell)$.

□

Thus we trapped the set of small values of $P_{1,\theta}$ into at most $C_3\,3^m$ intervals:

Lemma 20. $SSV(\theta,\ell\,m)\subset G(\theta)$.

This simple lemma cannot give us the power estimate $\le\frac{C}{n^c}$ for the Buffon needle landing 3^{-n}-near Sierpinski gasket. We need a much stronger lemma, where the LHS is much larger and the RHS is smaller.

Lemma 21. $SSV(\theta,\ell)\subset 3^{-m}\cup_{s=0}^{m}\omega_k$.

We postpone the proof of this lemma to Section 6.

Notations. Intervals $J(\theta_0)$ cover the compact $Q\cap\mathbb{R}\setminus 2W_m$. They are all of length 3^{-Bm}. Choose a finite subcover $\{J^t:=J(\theta_0^t), t=1,...,T\le 3^{Bm}\}$. Moreover we will think that J^t are half-open, half-closed intervals making *disjoint* cover of $Q\cap\mathbb{R}\setminus 2W_m$.

For each $t\le T$ we have at most $C_3\cdot 3^m$ distinct analytic functions $\lambda_{k,t}(\zeta), k=1,...,K(t)\le C_3\cdot 3^m$, in $D(\theta_0^t,2\cdot 3^{-Bm})$ which are zeros of $\phi_\zeta(z),\zeta\in D(\theta_0^t,2\cdot 3^{-Bm})$ in $[0,3^m]\times[-\delta_0/10,\delta_0/10]$.

We already considered

$$Y(k,t,\zeta):=\frac{1}{2}(\lambda_{k,t}(\zeta)+\bar\lambda_{k,t}(\bar\zeta))$$
$$y_s(k,t,\zeta):=3^{-s}Y(k,t,\zeta),\ s=\kappa,...,m$$
$$g_1(k,t,\zeta):=Y(k,t,\zeta)(c_2(\zeta)-c_1(\zeta))$$
$$g_2(k,t,\zeta):=Y(k,t,\zeta)(c_3(\zeta)-c_1(\zeta))\,.$$

Here κ is such that $Y(k,t,\theta) \in (3^{\kappa-1}, 3^{\kappa}]$.

We want to represent these numbers in a different, more convenient manner.

Let us consider (we skip index often ζ)

$$\Phi_{\zeta,m}(z) := \Phi_m(z) := \Pi_{s=0}^{m} \phi(3^{-s}\, z)\,.$$

Let $\mu_{i,t}(\zeta)$ be zeros of $\Phi_{m,\zeta}(z), \zeta \in D(\theta_0^t, 2 \cdot 3^{-Bm})$ in $[0, 3^m] \times [-\delta_0/10, \delta_0/10]$. When we run over all such μ_i's we obtain

$$\begin{aligned} Y(i,t,\zeta) &:= \frac{1}{2}(\mu_{i,t}(\zeta) + \bar{\mu}_{i,t}(\bar{\zeta})) \\ y(i,t,\zeta) &:= 3^{-m} Y(i,t,\zeta) \\ g_1(i,t,\zeta) &:= Y(i,t,\zeta)(c_2(\zeta) - c_1(\zeta)) \\ g_2(i,t,\zeta) &:= Y(i,t,\zeta)(c_3(\zeta) - c_1(\zeta))\,, \end{aligned}$$

we obtain all the previous numbers Y, y, g_1, g_2, only j's run over a different set, and we do not need subscript s to list all the y's.

We already proved

Lemma 22. $Y(i,t,\zeta) \geq a_1 \delta_0 \cdot 3^{-m}$, $y(i,t,\zeta) \geq a_1 \delta_0 \cdot 3^{-2m}$.

We need

Definition.

$$j(i,t,\theta) := (y(i,t,\theta) - 3^{-m} 3^{-\ell}, y(i,t,\theta) - 3^{-m} 3^{-\ell})\,.$$

Lemma 21 proves that

Lemma 23. *If* $\theta \in J^t$ *then* $SSV(\theta, \ell) \subset \cup_{i=1}^{C\, 3^m} j(i,t,\theta)$.

Lemma 24. *For any* $y \in j(i,t,\theta)$ *we have*

$$R(y(c_2(\theta) - c_1(\theta))) \leq C_4 R(y(i,t,\theta)(c_2(\theta) - c_1(\theta)))$$

and

$$R(y(c_3(\theta) - c_1(\theta))) \leq C_4 R(y(i,t,\theta)(c_3(\theta) - c_1(\theta)))\,.$$

Proof. We multiply the factors $r(3^s\, y)$ comprising R on the interval $j(i,t,\theta)$, where $s = m, m+1, ..., m+\ell$. We compare the product at an arbitrary point of the interval $j(i,t,\theta)$ to its value at the center. The length of the interval $j(i,t,\theta)$ is $3^{-m-\ell)}$, so the diffrence between the first factors in the LHS and the RHS is at most $C_5\, 3^{-m-\ell+m}$, and because both factors are bounded away from zero by 5/9 the ratio of the last factors in the LHS and RHS differs from 1 by at most $C_6\, 3^{-\ell}$. The second factors: the same but their ratio differs from 1 by at most $C_6\, 3^{\ell+1}$. We continue this comparison ℓ times. If we multiply all these ratios we get the convergent product and so at most (and at least) an absolute constant.

□

5. Some important standard lemmas

There are a few important lemmas which we have appealled to repeatedly. The first lemma, Lemma 25, uses the Carleson imbedding theorem. Its importance lies in its ability to establish a key relationship between the L^∞ norm of $f_{n,\theta}$ and the L^2 norm of $\widehat{f_{n,\theta}}$. This is because the Fourier transform changes the centers of intervals into the frequencies of an exponential polynomial.

The second lemma we will split into Lemmas 27, 28, and 29. They describe standard relationships between a holomorphic function, its zeroes, its boundary values, and its non-zero interior values. Because we use them so often, we have taken the trouble of stating and proving them so as to streamline the main argument of the paper.

5.1. In the spirit of the Carleson imbedding theorem.

Lemma 25. *Let $j = 1, 2, ...k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$. Then*

$$\int_0^1 |\sum_{j=1}^k c_j e^{i\alpha_j y}|^2 dy \le C\, k \cdot \sup_{I \text{ a unit interval}} \#\{A \bigcap I\}\,.$$

Proof. Let $A_1 := \{\mu = \alpha + i : \alpha \in A\}$. Let $\nu := \sum_{\mu \in A_1} \delta_\mu$. This is a measure in $\mathbb{C}_+$. Obviously its Carleson constant

$$\|\nu\|_C := \sup_{J \subset \mathbb{R},\, J \text{ is an interval}} \frac{\nu(J \times [0, |J|])}{|J|}$$

can be estimated as follows

$$\|\nu\|_C \le 2 \sup_{I \text{ a unit interval}} \text{card}\{A \bigcap I\}\,. \tag{5.1}$$

Recall that

$$\forall f \in H^2(\mathbb{C}_+) \int_{C_+} |f(z)|^2\, d\nu(z) \le C_0\, \|\nu\|_C \|f\|_{H^2}^2\,, \tag{5.2}$$

where C_0 is an absolute constant. Now we compute

$$\int_0^1 |\sum_{j=1}^k c_j e^{i\alpha_j y}|^2 dy \le e^2 \int_0^1 |\sum_{j=1}^k c_j e^{i(\alpha_j+i)y}|^2 dy \le$$

$$e^2 \int_0^\infty |\sum_{j=1}^k c_j e^{i(\alpha_j+i)y}|^2 dy = e^2 \int_{\mathbb{R}} |\sum_{\mu\in A_1} \frac{c_\mu}{x-\mu}|^2\,,$$

where $c_\mu := c_j$ for $\mu = \alpha_j + i$. The last equality is by Plancherel's theorem.

We continue

$$\int_{\mathbb{R}} |\sum_{\mu\in A_1} \frac{c_\mu}{x-\mu}|^2 = \sup_{f\in H^2(C_+),\,\|f\|_2\le 1} \left|\langle f, \sum_{\mu\in A_1} \frac{c_\mu}{x-\mu}\rangle\right|^2 =$$

$$4\pi^2 \sup_{f\in H^2(C_+),\,\|f\|_2\le 1} |\sum_{\mu\in A_1} c_\mu f(\mu)|^2 \le C\,\#\{A_1\} \sup_{f\in H^2(C_+),\,\|f\|_2\le 1} \sum_{\mu\in A_1} |f(\mu)|^2 \le$$

$$C\,\#\{A\} \sup_{f\in H^2(C_+),\,\|f\|_2\le 1} \int_{C_+} |f(z)|^2\, d\nu(z) \le 2C_0 C\,\#\{A\} \sup_{I \text{ a unit interval}} \#\{A \bigcap I\}\,.$$

This is by (5.5) and (5.1). The lemma is proved.

□

Now we are going to prove a stonger assertion by a simpler approach. This stronger assertion is what is used in the main part of the article.

Lemma 26. *Let $j = 1, 2, ...k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$. Then Suppose*

$$\int_{\mathbb{R}} (\sum_{\alpha\in A} \chi_{[\alpha-1,\alpha+1]}(x))^2\, dx \le S\,, \tag{5.3}$$

Then there exists an abolute constant C

$$\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i\alpha y}|^2\, dy \le C\, S\,. \tag{5.4}$$

Remark. Lemma 26 is obviously stronger than Lemma 25. In fact, let S_0 be the maximal number of points A in any unit interval. Then $f(x) := \sum_{\alpha\in A} \chi_{[\alpha-1,\alpha+1]}(x) \le 2S_0$. Now $\int_{\mathbb{R}} f^2(x)dx \le 4kS_0$, where k as above is the cardinality of A. We can put now $S := 4kS_0$, apply Lemma 26 and get the conclusion of Lemma 25. The proof of Lemma 26 does not require the Carleson imbedding theorem. Here it is.

Proof. Using Plancherel's theorem we write

$$\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i\alpha\, y}\, dy|^2 \le e\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i(\alpha+i)\, y}\, dy|^2 \le e\int_0^\infty |\sum_{\alpha\in A} c_\alpha e^{i(\alpha+i)\, y}\, dy|^2 =$$

$$e\int_{\mathbb{R}}\left|\sum_{\alpha\in A}\frac{c_\alpha}{\alpha+i-x}\right|^2 dx\,.$$

Recall that

$$H^2(\mathbb{C}_+) \text{ is orthogonal to } \overline{H^2(\mathbb{C}_+)} \tag{5.5}$$

Now we continue

$$\int_{\mathbb{R}}\left|\sum_{\alpha\in A}\frac{c_\alpha}{\alpha+i-x}\right|^2 dx \le$$

$$\int_{\mathbb{R}}\left|\sum_{\alpha\in A}\frac{c_\alpha}{\alpha+i-x}-\sum_{\alpha\in A}\frac{c_\alpha}{\alpha-i-x}\right|^2 dx =$$

$$\frac{\pi}{2}\int_{\mathbb{R}}\left|\sum_{\alpha\in A} c_\alpha P_1(\alpha-x)\right|^2 dx\,,$$

where P_1 is the Poisson kernel in the half-plane C_+ at hight $h=1$:

$$P_h(x) := \frac{1}{\pi}\frac{h}{h^2+x^2}\,.$$

We continue by noticing that $P_1\star\chi_{[\lambda-1,\lambda+1]}(x) \ge c\,P_1(\lambda-x)$ with absolute positive c. This is an elementary calculation, or, if one wishes, the Harnack's inequality. Now we can continue

$$\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i\alpha\, y}\, dy|^2 \le \frac{\pi e}{2c}\int_{\mathbb{R}}\left|(P_1\star\sum_{\alpha\in A} c_\alpha\chi_{[\alpha-1,\alpha+1]})(x)\right| dx\,.$$

Now we use the fact that $f\to P_1\star f$ is a contraction in $L^2(\mathbb{R})$. So

$$\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i\alpha\, y}\, dy|^2 \le \frac{\pi e}{2c}\int_{\mathbb{R}} |\sum_{\alpha\in A} c_\alpha\chi_{[\alpha-1,\alpha+1]}(x)|\, dx \le C\,S\,.$$

The lemma is proved.

□

5.2. **A Blaschke estimate.**

Lemma 27. *Let D be the closed unit disc in $\mathbb{C}$. Suppose ϕ is holomorphic in an open neighborhood of D, $|\phi(0)| \geq 1$, and the zeroes of ϕ in $\frac{1}{2}D$ are given by $\lambda_1, \lambda_2, ..., z_M$. Let $C = ||\phi||_{L^\infty(D)}$. Then $M \leq log_2(C)$.*

Proof. Let

$$B(z) = \prod_{k=1}^{M} \frac{z - \lambda_k}{1 - \bar{\lambda_k} z}.$$

Then $|B| \leq 1$ on D, with $=$ on the boundary. If we let $g := \frac{\phi}{B}$, then g is holomorphic and nonzero on $\frac{1}{2}D$, and $|g(e^{i\theta})| \leq C \; \forall \theta \in [0, 2\pi]$. Thus $|g(0)| \leq C$ by the maximum modulus principle. So we have

$$C \geq |g(0)| = \frac{|\phi(0)|}{|B(0)|} \geq \prod_{k=1}^{M} \frac{1}{|\lambda_k|} \geq 2^M.$$

□

Lemma 28. *In the same setting as Theorem 27, the following is also true for all $\delta \in (0, 1/3)$: $\{z \in \frac{1}{4}D : |\phi| < \delta\} \subseteq \bigcup_{1 \leq k \leq M} B(\lambda_k, \varepsilon)$, where*

$$\varepsilon := \frac{9}{16}(3\delta)^{1/M} \leq \frac{9}{16}(3\delta)^{1/log_2(C)}.$$

Proof. Let $\delta \in (0, 1/3)$, and let $z \in \frac{1}{4}D$ such that $|z - \lambda_k| > \varepsilon \; \forall k$. Note that g is harmonic and nonzero on $\frac{1}{2}D$ with $|g(0)| \geq 2^M$. Thus Harnack's inequality ensures that $|g| \geq \frac{1}{3}2^M$ on $\frac{1}{4}D$, so there

$$|\phi(z)| \geq |g(z)B(z)| \geq \frac{1}{3}2^M \prod_{k=1}^{M} |\frac{z - \lambda_k}{1 - \bar{\lambda_k} z}| \geq (\frac{16\varepsilon}{9})^M \frac{1}{3} = \delta.$$

We can conclude the proof by the contrapositive. □

Lemma 29. *Let $\delta \in (0, 1/3)$. Let ϕ be holomorphic in the horizontal strip $\mathbb{R} \times (-14H, 14H)$, with H large enough, and with $|\phi| \leq C$ in the strip. Let $Box = [x-1, x+1] \times [-H, H]$. Let $max_{z \in Box}|\phi| \geq 1$. Let ε be as in Theorem 28, and call the zeroes of ϕ in an ε neighborhood of Box by the names $\lambda_1, \lambda_2, ..., \lambda_M$. Then $M \leq log_2(C)$, and*

$$\{z \in Box : |\phi(z)| \leq \delta\} \subseteq \bigcup_{1 \leq k \leq M} B(\lambda_k, \varepsilon).$$

Proof. Take D to be a disc of radius $12\,H$ centered at the z which maximizes $|\phi|$ in Box. □

6. Analytic tiling

6.1. **Discussion.** Let $c_1 = \cos(\pi/2 - \theta)\,,\ c_2 = \cos(7\pi/6 - \theta)\,,\ c_3 = \cos(\pi/6 + \theta)$. Let aslo

$$\phi(z) := \frac{1}{3}(e^{ic_1 z} + e^{ic_2 z} + e^{ic_3 z})\,.$$

$$f_m(z) := \Pi_{k=0}^{m}\phi(3^k z)\,.$$

$$\Phi(z) := \Pi_{k=0}^{+\infty}\phi(3^{-k} z)\,.$$

Notice that for 3 particular angles θ, f_m COLLAPSES. By this I mean that being multiplied through it becomes 3^{-m} times the sum of exponents e^{ikz}, where k just goes from 0 to 3^m (something like that) with step 1. It is called COLLAPSING because now this $f_m(z)$ becomes $\frac{e^{i3^m z}-1}{3^m(e^{iz}-1)}$. The zeros are clear: these are zeros of $\sin\frac{1}{2}3^m z$, and the set of small values SSV:= $\{x \in [-1,1] : |f_m(x)| < 3^{-m}\cdot 3^{-Am}\}$ is contained in at most $C\,3^m$ intervals of length $c\,3^{-m}\cdot 3^{-Am}$ each. Let us rescale by changing variable $\zeta = 3^m z$. Then $F_m(\zeta) := f_m(3^{-m}\zeta)$. We can write that

$$SSV(F_m) := \{x \in [-3^m, 3^m] : |F_m(x)| < 3^{-(A+1)m}\}$$

is contained in at most $C\,3^m$ intervals of length $c\cdot 3^{-Am}$ each. It is a crucial property, let us call it (Consequence of Colapsing), and it is valid for the function

$$F_m(\zeta) := \Pi_{k=0}^{m}\phi(3^{-k} z)\,,$$

which neatly converges to

$$\Phi(z) = \Pi_{k=0}^{+\infty}\phi(3^{-k} z)\,.$$

Recall that this property (Consequence of Colapsing) so far happened for three particular values of θ. And why? Because of combinatorial thing: tiling in these three directions.

So (Consequence of Colapsing) is the property of entire function's set of small values to be covered by ceratin number of discs of a certain radius.

But the same happened for product Cantor sets in ANY direction if we assume the property assumed in Laba–Zhai's paper! For 1/4 Cantor set, for example, (Consequence of Colapsing) happens in any θ because there is tiling in one direction. More precisely, because the product of $\cos(4^k\cdot y), k = 0, ..., m$ IS ALMOST $\sin 4^m\cdot y$. The formula saying that the product of cosines with such frequencies is ALMOST just one single sine gives us that the (Consequence of Colapsing) happens in each direction.

What if this combinatorial collapsing is superflous? I mean what if function $\Phi(z)$ has the needed property ALWAYS, sort of property of its amost periodic zeros? More precisely what if the following claim is true for any real c_1, c_2, c_3:

$$SSV(\Phi) := \{x \in [-3^m, 3^m] : |\Phi(x)| < 3^{-(A" \cdot B)m}\}$$

is contained in at most $C\, 3^{25m}$ intervals of length $c \cdot 3^{-Bm}$ each. Quantors should be like this $\exists d$ such that $\forall B \exists A"$ such that $SSV = \{|\Phi| < 3^{-A" Bm}\}$ is contained in at most 3^{dm} intervals of length at most 3^{-Bm}.

This would follow if the following elementary fact would be true:

Let g be a bounded analytic function in the unit disc with at most N zeros and such that $|g| \leq 1, |g(0)| \geq 1/2$. Then there exists d such that for any B there exists A such that $\{z : |z| < (1 - \frac{1}{N}), |g(z)| \leq \frac{1}{N^{AB}}\}$ is covered by at most N^d euclidean discs of radius $\frac{1}{N^B}$.

The fact above is probably false. But for a special type of function g with a certain structure we proved it in Section 6. The same tipe of the claim must be true for more complicated products.

6.2. **Set of small values is contained in the right number of correctly sized intervals.** Let c_1, c_2, c_3 be *arbitrary* real numbers such that $|c_i| \leq 1$. Let

$$\phi(z) = \frac{1}{3}(e^{ic_1 x} + e^{ic_2 x} + e^{ic_3 x})\,.$$

$$\Phi(z) := \Pi_{k=0}^{\infty} \phi(3^{-k} z)\,.$$

$$\Psi_m(z) := \Pi_{k=0}^{m} \phi(3^k z)\,.$$

Recall that by box we understand B_{x_0}, the unit square centered at $x_0 \in R$. Given a box B_{x_0} we consider also the collection of *boxiks*: the images of B_{x_0} by the maps $z \to 3^{-k} z$, $k = 1, ..., m$. Boxik $B_{x_0,k}$ is the square centered at $3^{-k} x_0$ and of size $3^{-k}, k = 1, ..., m$. The union of the boxiks is called the tail of the box B_{x_0}. The union of the box B_{x_0} and its tail is called box-with-tail.

Recall that we proved an easy claim

Theorem 30. *In each box function ϕ has at most M zeros, where M is an absolute constant. Moreover, there exists absolute constant M such that for any ℓ the set $\{z : |\Im z| \leq 1, |\phi(z)| < 3^{-M\ell}\}$ is contained in discs of radius $3^{-\ell}$ centered at zeros of ϕ.*

It is a particular case of Tijdeman's theorem, [?], [?], which claims that for every sum of exponentilas $e^{ic_\ell z}$ consisting of L terms, the number of zeros for this sum in any disc of radius R is bounded by $3L + ARc$, where A is an absolute constant, $c := \max(|c_1|, ..., |c_L|)$.

Now we want to prove that the same result (with different M of course, but also absolute) is valid for Φ.

Theorem 31. *Let ϕ, Φ be as above with real exponents such that $|c_i| \le 1$. There exists an absolute constant K, such that in each box function Φ has at most K zeros. In the language of ϕ this means that in each box-with-tail function ϕ contains at most K of its zeros.*

Proof. The crucial simple observation is the following: let x_1, x_2, x_3 be reals, such that $e^{ix_j}, j = 1, 2, 3$ are located counterclockwise on the circle, let $e^{ix_1} + e^{ix_2} + e^{ix_3} = 0$; then $x_1 - x_2, x_2 - x_3, x_3 - x_1 = \frac{2\pi}{3} \mod 2\pi$.

Moreover this claim is stable, namely: let x_1, x_2, x_3 be reals, such that $e^{ix_j}, j = 1, 2, 3$ are located counterclockwise on the circle, let $\frac{1}{3}|e^{ix_1} + e^{ix_2} + e^{ix_3}| < 3^{-k}$; then

$$|x_1 - x_2 - \frac{2\pi}{3}|, |x_2 - x_3 - \frac{2\pi}{3}|, |x_3 - x_1 - \frac{2\pi}{3}| < D \cdot 3^{-k} \tag{6.1}$$

with absolute D, if all this differences are taken $\mod 2\pi$.

Now consider the boxiks $B_{x_0,j}, j = m, ..., 1$. Let $B_{x_0,k}$ contain a zero (call it z_k) of ϕ. Boxik has size 3^{-k} and $|\phi'(z)| \le A$ (absolute constant) in $|\Im z| \le 1$, so having a zero of ϕ in $B_{x_0,k}$ forces

$$|\phi(x)| \le A \cdot 3^{-k} \; \forall x \in B_{x_0,k} \cap \mathbb{R}\,.$$

Then we can apply (6.1) to $x_i := c_i x, x \in B_{x_0,k} \cap \mathbb{R}$. Then we get from (6.1)

$$|e^{i3^s(c_1 x - c_2 x)} - 1|, |e^{i3^s(c_3 x - c_2 x)} - 1| < D' \cdot 3^{-(k-s)} << 1 \; \forall s = 1, 2, ..., m - C\,. \tag{6.2}$$

Here D', C are absolute constants. Now, (6.2) implies

$$|1 - \phi(x)| < D' \cdot 3^{-(k-s)} << 1 \; \forall s = 1, 2, ..., m - C \; \forall x \in B_{x_0,k-s} \cap \mathbb{R}\,. \tag{6.3}$$

The size of $B_{x_0,k-s}$ is $3^{-(k-s)}$, so we conclude from (6.3) and from the boundedness of the deriviative of ϕ in the strip that

$$|1 - \phi(z)| < D' \cdot 3^{-(k-s)} << 1 \; \forall s = 1, 2, ..., m - C \; \forall x \in B_{x_0,k-s}\,. \tag{6.4}$$

So if k is the largest index of a boxik containing a zero of ϕ the boxiks with index between k and m do not contain zeros by maximality of index, and boxiks with index between $k+1$ and C do not contain zeros because of (6.4). Zeros of ϕ thus can be only in $B_{x_0,k} \cup B_{x_0,C} \cup ...B_{x_0,1} \cup B_{x_0}$, and by Theorem 30, the number of zeros of ϕ in the box-with-tail is bounded by $(C+2)M$. Theorem 31 is proved. $\square$

The next result is about covering the set of small values of function Φ. It shows that in this respect $\Phi(z)$ behaves almost like $\sin z$.

$$SSV_m(\Phi, \ell) := \{x \in [-3^m, 3^m] : |\Phi(x)| < 3^{-\ell}\}\,.$$

Theorem 32. *There exists two absolute constants B_0, C such that for any $\ell \geq m$ and for any A there exista at most $c \cdot 3^m$ discs or radius $3^{-A\ell}$ each whose union covers $SSV_m(\Phi, B_0 A\ell)$.*

Before giving the proof, the discussion: by Theorem 31 we have at most K zeros of Φ in each box B_{x_0}, in particular, for each center $x_0 \in [3^{m-1}, 3^m]$. Consider a disc of radius $3^{-A\ell}$ centered at each of this zeros. Morally we tend to think that in B_{x_0} but outside of these at most K discs $|\Phi(z)| \geq (3^{-A\ell})^K$. That would be the end of the story, but unfortunately, to substantiate the claim we need two things

$$|\Phi(z)| \leq C_0^m\,, \forall z \in B_{x_0}\,, \tag{6.5}$$

with absolute constant C_0, and

$$\exists z_0 \in B_{x_0}\ |\Phi(z_0)| \geq 3^{-c_0 m}\,, \tag{6.6}$$

with absolute positive c_0.

If we have (6.5) and (6.6) then, in fact, in $B_{x_0} \setminus \cup_{k+1}^K B(z_i, 3^{-A\ell}$ one has

$$|\Phi(z)| \geq e^{-c_1(c_0+C_0)m - c_2 K A\ell} \geq e^{-B_0\ell}$$

(with absolute c_1, c_2) as required. The latter estimate is just the repetition of an elementary consideration in [5] based on Harnack's inequality.

Of course (6.5) is obvious because $|\phi(z)| \leq C_0$ in a unit strip around $\mathbb{R}$, and because of the fast convergence of product forming Φ.

We are left to prove (6.6).

Proof. Consider a box B_{x_0} for $x_0 \in [3^{m-1}, 3^m]$. Among its boxiks $B_{x_0,k}, k = 1, ..., m$ cross those for which $2B_{x_0,k}$ contains a zero of ϕ. The number of crossed boxiks is at most K. Let x be a real point in box-with-tail built for B_{x_0}. The orbit of x is the union of $3^j x$, where we get one of such points in each boxik. So j can be positive or negative, each orbit consists of $m+1$ point. In each of them we can find the exceptional set of reals of measure at most 3^{-100m} where ϕ is very small. Outside of it in the rest of crossed boxiks reals $|\phi(x)| \geq 3^{-100Mm}$. This is by Theorem 30. If we are in a boxik, which was not crossed, we remember that its size is at least 3^{-m}, and we are so at least 3^{-m}-far from any zero of ϕ. Then again $|\phi(x)| \geq 3^{-100Mm}$. This is again by Theorem 30. Consider the orbits of all exceptional sets of all crossed boxiks. Call it E. It is still such a small set, that $|\{x \in Boxik : x \notin E\} \geq 0.9$ size of the boxik.

Consider $B := B_{x_0,k}, k = 1, ..., m$ with the largest index, which is not crossed. It can be $B_{x_0,m}$ or $B_{x_0,m-1}$. Choose $x \in B \cap \mathbb{R} \setminus E$. Suppose $|\phi(x)| \in (3^{-k_1-1}, 3^{-k_1}]$. By what was just said $k_1 \leq 100Mm$. Let us consider two cases. If $k_1 \leq 2m$ we call k_1 "light". Otherwise it is called "heavy". If $k_1 \leq C$ (absolute), we come to $3x$ in the next boxik and find k_2 (light or heavy) for it.... If $k_1 > C$, but $k_1 < 2m$ we notice that by (6.1)

$$|1 - \phi(3x)|, ..., |1 - \phi(3^{k_1/2}x| < D' 3^{-k_1/2} . \tag{6.7}$$

If $k_1 > 2m$ (heavy case) we can stop. All values of ϕ on the orbit of x are either large by (6.7) or bigger than 3^{-100Mm} for finitely many orbit points lying in the finitely many ($\leq K$) crossed boxes. At any rate $\Phi(x)| \geq c^m \cdot 3^{-100KMm}$. This is (6.6).

If $C < k_1 < 2m$ we find k_2 as follows. Consider point $3^{k_1/2+1}x$ and find k_2 such that $|\phi(3^{k_1/2+1}x)| \in (3^{-k_2-1}, 3^{-k_2}]$. If $k_2 \leq C$ (absolute), we come to $3 \cdot 3^{k_1/2+1}x$ in the next boxik and find k_3 for it.... If $k_2 > C$, but $k_2 < 2m$ we notice that by (6.1)

$$|1 - \phi(3 \cdot 3^{k_1/2+1}x)|, ..., |1 - \phi(3^{k_2/2} \cdot 3^{k_1/2+1}x| < D' 3^{-k_2/2} . \tag{6.8}$$

This algorithm ends after finitely many steps (at most m steps). Only one k_i, and only the last one, can be heavy. Notice that, the sum of all light k_j is at most $3m$. When our algorithm stops on s-th step, we can write

$$k_1 + ... + k_s \leq 3m + 100Mm \, , s \leq m \, .$$

$$|\Phi(x)| \geq 3^{-100KMm} 3^{-k_1-k_2-...-k_s} 3^{-s} c^m \geq 3^{-B_0 m} \, .$$

So (6.6) is proved together with Theorem 32.

□

7. Combinatorial theorem

Theorem 33. *Let* $\theta \in \{\theta : |\{x : f_N^*(x) := \sup_{n\le N} f_{n,\theta}(x) \le K\}| \le \frac{1}{K^3}\}$. *Then*

$$\max_{n:0\le n\le N} \|f_{n,\theta}\|^2_{L^2(\mathbb{R})} \le C\,K\,.$$

To prove this we first need the following claim, which is the main combinatorial assertion of this article. It repeats the one in [15] but we give a slightly different proof.

We fix a direction θ, we think that the line ℓ_{theta} on which we project is $\mathbb{R}$. If $x \in \mathbb{R}$ then by N_x we denote the line orthogonal to $\mathbb{R}$ and passing through point x, we call N_x a needle. By F_L we denote $\{x \in \mathbb{R} : f_N^*(x); = \max_{0\le n\le N} f_{n,\theta}(x) > L\}$.

Theorem 34. *There exists an absolute constant* C *such that for any large* K *and* M

$$|F_{4KM}| \le C\,K\,|F_K| \cdot |F_M|\,. \tag{7.1}$$

Proof. This will be a proof by greedy algorithm. First choose $y \in F_{4K}$ and consider needle N_y and triangles of certain size $3^{-j_y}, j_y \le N$ intersecting N_y. Consider any family of this sort having more than $4K$ elements. Fix such a family. We will "fathorize" it, i.e. we consider the father of each element in the family. Two things may happen: 1) there are more than $4K$ distinct fathers; 2) number of fathers is at most $4K$. In the latter case the number of fathers is at least $2K$. In fact, we slash the number of elements by fathorizing, but oit more than by factor of 1/2. If the first case happens fathorize again, do this till we get to the second case.

After doing this procedure with all $x \in F_{4K}$ and all families of cardinality bigger than $4K$ of equal size triangles intersecting needle N_x we come to some awfully complicated set of triangles. But we will consider now maximal-by-inclusion triangles of this family, the family of these maximal triangles is called $\mathcal{F}_0$.

Choose triangle $Q_{00} \in \mathcal{F}_0$ such that its sidelength $\ell(Q_{00})$ is maximal possible in $\mathcal{F}_0$. It is very important to notice that $\mathcal{F}_0$ contains at least $@K-1$ triangles of the same size as Q_{00} pierced by a needle N_{y_0}. This is because of maximality of the lengthsize, the stack pierced by N_{y_0} could not be eaten up even partially by bigger in size triangles from some other stack. So let us call by $Q_{01}, ..., Q_{02K-1}, ..., Q_{0S}$,

$S \geq 2K - 1$. They are of the same size as Q_{00} and all intersect a certain needle N_{y_0}.

Denote

$$I_0 = \text{proj}\, Q_{00}\,.$$

Consider all $q \in \mathcal{F}_0$ such that

$$\text{proj}\, q \cap 20\, I_0 \neq \emptyset\,.$$

Call them $\mathcal{F}(Q_{00})$. Of course $\ell(q) \leq \ell(Q_{00})$. For every such q consider a Cantor square Q, $q \subset Q$, such that $\ell(Q) = \ell(Q_{00})$. Such Q's form family $\tilde{\mathcal{F}}(Q_{00})$.

Lemma 35. *For every $y \in \mathbb{R}$ the needle N_y intersects at most $4K$ triangles of the family $\tilde{\mathcal{F}}(Q_{00})$.*

Proof. Suppose contrary. Then N_y intersects more than $4K$ of triangles from $\tilde{\mathcal{F}}(Q_{00})$. So $y \in F_{4K}$, and our pierced family is one of those which we considered at the begining. It can be fathorized. Then the square of size $\geq 2\,\ell(Q_{00})$ will be prrsent in $\mathcal{F}_0$. Contradiction with maximality of length.

□

Lemma 36. *card*$\tilde{\mathcal{F}}(Q_{00}) \leq 88\, K$.

Proof.

$$\text{card}\, \tilde{\mathcal{F}}(Q_{00}) \cdot \ell(Q_{00}) = \sum_{Q \in \tilde{\mathcal{F}}(Q_{00})} \ell(Q) \leq$$
$$\int_{22 I_0} \text{card}\, \{Q \in \tilde{\mathcal{F}}(Q_{00}) : Q \cap N_y \neq \emptyset\}\, dy \leq$$
$$4K \cdot 22\ell(Q_{00})\,.$$

This is by Lemma 35.

□

Lemma 37. *There exists an interval $J_0 \subset I_{y_0}$ such that $|J_0| \geq c \cdot |I_0|$ with a ceratin absolute positive c. And $J_0 \subset F_K$.*

Proof. We already noticed that $Q_{00}, Q_{01}, ..., Q_{02K-1}$ intersect needle N_{y_0}. Then at least half of them have their center of symmetry to the right of N_{y_0}, or at least half of them have their center of symmetry to the left of N_{y_0}. Assume that the first case occurs. Then the segment $[y_0, c \cdot \ell(Q_{00})]$ obviously is contained in F_K.

□

Lemma 38. $|F_{4KM} \cap 20I_0| \leq C\,K\,\ell(Q_{00}) = C\,K\,|I_0|$.

Proof. Of course $F_{4KM} \subset F_{4K}$. For $y \in F_{4KM} \cap 20I_0$ the whole family of small triangles whose quantity is $> 4KM$ intersecting N_y will be inside one of those $Q \in \tilde{\mathcal{F}}(Q_{00})$, whose number is at most $88K$ by Lemma 36. Let us enumerate $Q^1, ..., Q^s$, $s \leq 88K$ elements of $\tilde{\mathcal{F}}(Q_{00})$. So there exists $i = 1, ..., s$ such that

$$y \in \text{dilated copy of}\, F_M \,\text{in}\, \text{proj}\, Q^i\,.$$

Hence

$$F_{4KM} \cap 20I_0 \subset \cup_{i=1}^{88K} \text{dilated copy of}\, F_M \,\text{in}\, \text{proj}\, Q^i\,.$$

So

$$|F_{4KM} \cap 20I_0| \leq \sum_{i=1}^{88K} \ell(Q^i)|F_M| \leq 88K\,\ell(Q_{00})|F_M|\,.$$

□

Lemma 39. $|F_{4KM} \cap 20I_0| \leq 88c^{-1}K|F_m| \cdot |J_0|$.

Now we want to repeat all steps for $F^0_{4K} := F_{4K} \setminus 20I_0$. So we fathorize triangles peirced by needles N_x, $x \in F^0_{4K}$. As before we get families $\mathcal{F}_1$, maximal sidelength trinagle Q_{11}, families $\mathcal{F}(Q_{11})$, $\tilde{\mathcal{F}}(Q_{11})$. Notice that $\mathcal{F}_1 < \mathcal{F}_0$ in the sense that for every $q \in \mathcal{F}_1$ there exists $q \in \mathcal{F}_0$ such that q is contained in Q. It is also clear that

$$\ell(Q_{11}) \leq \ell(Q_{00})\,.$$

Obviously $Q_{00}, Q_{01}, ...$ are not in $\mathcal{F}_1$, their projections even do not intersect $\mathbb{R}\setminus 20I_0$.

There are at least $2K-1$ brothers of Q_{11}: $Q_{12}, ..., Q_{12K-1}, ...$ in $\mathcal{F}_1$ such that they are of the same size $\ell(Q_{11})$ and they (and Q_{11}) intersect the same needle N_{y_1}, $y_1 \in \mathbb{R} \setminus 20I_0$. This is again the maximality of the sidelength among $\mathcal{F}_1$ triangles. Let $I_1 := \text{proj}\, Q_{11}$. Notice that

$$I_1 \cap I_0 = \emptyset\,.$$

In fact, $y_1 \in I_1, y_1 \notin 20I_0$, Q_{11} size is much smaller than $20|I_0|$. We consider all $q \in \mathcal{F}_1$ such that

$$\text{proj}\, q \cap (20I_1 \setminus 20I_0) \neq \emptyset\,.$$

Call this family $\mathcal{F}(Q_{11})$. For every $q \in \mathcal{F}(Q_{11})$ consider Cantor triangle Q containing q and of the size $\ell_1 = \ell(Q_{11})$. Maximal-by-inclusion among such Q's form $\tilde{\mathcal{F}}(Q_{11})$.

Lemma 40. *For any $y \in R \setminus 20I_0$, N_y intersects at most $4K$ triangles of $\tilde{\mathcal{F}}(Q_{11})$.*

Proof. Suppose contrary. Then there exists $y_1' \in F_{4K} \cap (\mathbb{R} \setminus 20I_0)$, and a subfamily of $\tilde{\mathcal{F}}(Q_{11})$ of cardinality bigger than $4K$ intersects $N_{y_1'}$. It can be fathorized. Then triangles of size $\geq 2\ell(Q_{11})$ would belong to $\mathcal{F}_1$. This contradicts the maximality of $\ell(Q_{11})$.

□

Lemma 41. *For any $z \in \mathbb{R}$, N_z intersects at most $8K$ triangles of $\tilde{\mathcal{F}}(Q_{11})$.*

Proof. Suppose contrary. Then there exists $z \in F_{4K}$, and a subfamily of $\tilde{\mathcal{F}}(Q_{11})$ of cardinality bigger than $4K$ intersects N_z. Now there is an end-point of $20I_1 \setminus 20I_0$ (call it a), which is closest to z. Let it be on the right of z. Then another end-point is also on the right but farther away. As every traingle from the family has a) z in its projection, and b) a ceratin point to the right of a in its projection (their projections intersect $20I_1 \setminus 20I_0$–by definition), then all of them have a in its projection. Let us be lavish and say that 50 percent of them have a in their projection (the fact is that it is not lavishness, it is necessity: next step will be to consider in the future $20I_2 \setminus (20I_0 \cup 20I_1)$, and their can be 2 closest points to z: one on the left, say, b, and one on the right, say, a, and we can guarantee that 50 percent of our triangles have either b or a in their projections simultaneously). We use the previous Lemma 40, and get that this 5) percent is $\leq 4K$. So we are done.

□

Lemma 42. *card* $\tilde{\mathcal{F}}(Q_{11}) \leq 172K$.

Proof.

$$\text{card}\, \tilde{\mathcal{F}}(Q_{11}) \cdot \ell(Q_{11}) = \sum_{Q \in \tilde{\mathcal{F}}(Q_{11})} \ell(Q) \leq$$

$$\int_{22I_1} \text{card}\, \{Q \in \tilde{\mathcal{F}}(Q_{11}) : Q \cap N_y \neq \emptyset\}\, dy \leq$$

$$8K \cdot 22\ell(Q_{11})\,.$$

This is by Lemma 35.

□

Lemma 43. *There exists an interval $J_1 \subset I_1$, $|J_1| \leq c \cdot |I_1|$, such that $J_1 \subset F_K$.*

Proof. The same proof as for Lemma 37. □

Lemma 44. $|F^0_{4KM} \cap 20 I_1| \le C\, K\, \ell(Q_{11} \le C|, K\, |I_1|\,.$

Proof. The same proof as for Lemma 38. □

Combining Lemmas 43, 44 we get

Lemma 45. $|F^0_{4KM} \cap 20 I_1| \le C\, c^{-1}\, K\, |J_1|\,.$

We continue by introducing

$$F^1_{4KM} = F_{4KM} \setminus (20 I_0 \cup 20 I_1)\,.$$

We repeat the whole procedure. There will be I_2, $J_2 \subset I_2 \cap F_K, |J_2| \ge c \cdot |I_2|$:

$$I_2 \cap (I_1 \cup I_0) = \emptyset\,,$$

$$|F_{4KM} \cap 20 I_2| \le C c^{-1} K |J_2| |F_M|\,,$$

et cetera.

Finally,

$$|F_{4KM}| \le |F_{4KM} \cap 20 I_0| + |(F_{4KM} \setminus 20 I_0) \cap 20 I_1| + ... + |(F_{4KM} \setminus 20 I_0 \cup 20 I_1 \cup .... 20 I_{j-1}) \cap 20 I_j| + ... \le$$

$$C'\, K\, |F_M| \sum_{j=0}^{\infty} |J_j| \le C'\, K\, |F_M|\, |F_K|\,.$$

We are done with Theorem 34. □

Now we can prove Theorem 33.

Proof. Let $E_j := \{x : f_{n,\theta}(x) > (4K)^{j+1}\}$, $j = 0, 1, ...$ We know by Theorem 34 that

$$|E_j| \le (CK)^j |E_0|^{j+1}\,.$$

Hence,

$$\int f_{n,\theta}(x)^2\, dx \le 4K \int f_{n,\theta}(x)\, dx + \sum_{j+0}^{\infty} \int_{E_j \setminus E_{j+1}} f_{n,\theta}(x)^2\, dx \le$$

$$4CK + (4K)^{j+2}\, (CK)^j |E_0|^{j+1}\,.$$

If $|\{x : f^*_N(x) > K\}| \le 1/K^{2+\tau}$ then for all $n \le N$ we can immediately read the previous inequality as

$$\int f_{n,\theta}(x)^2\, dx \le C(\tau)\, K\,.$$

□

References

1. M. Bateman, A.Volberg, *An estimate from below for the Buffon needle probability of the four-corner Cantor set*, arXiv:math. 0807.2953v1, 2008, pp. 1-11.
2. M. Bateman, N.Katz, *Kakeya sets in Cantor directions*, arXiv:math. 0609187v1, 2006, pp. 1–10.
3. M. Bateman, *Kakeya sets and the directional maximal operators in the plane*, arXiv:math.CA 0703559v1, 2007, pp. 1–20.
4. A. S. Besicovitch, *Tangential properties of sets and arcs of infinite linear measure*, *Bull. Amer. Math. Soc.* **66** (1960), 353–359.
5. M. Bond, A. Volberg: *Buffon needles landing near Sierpinski gasket*, arXiv:0905.0207v2, 10 june 2009, pp. 1–28.
6. M. Bond, A. Volberg: *Estimates from below of the Buffon noodle probability for undercooked noodles*, arXiv:math/0811.1302v1, 2008, pp. 1–10.
7. K. J. Falconer, The geometry of fractal sets. Cambridge Tracts in Mathematics, 85. C.U.P., Cambridge–New York, (1986).
8. U. Keich, On L^p bounds for Kakeya maximal functions and the Minkowski dimension in $\mathbb{R}^2$, *Bull. London. Math. Soc.* **31** (1999), pp. 213–221.
9. R. Kenyon, *Projecting the one-dimensional Sierpinski gasket, Israel J. Math.* **97** (1997), 221–238.
10. I. Laba, K. Zhai, *Favard length of product Cantor sets*, arXiv:0902:0964v1, Feb. 5 2009.
11. J. C. Lagarias and Y. Wang, *Tiling the line with translates of one tile*, *Invent. Math.***124** (1996), 341–365.
12. P. Mattila, *Orthogonal projections, Riesz capacities and Minkowski content*, *Indiana Univ. Math. J.* **39** (1990), 185–198.
13. P. Mattila, *Hausdorff dimension, projections, and the Fourier tarnsform*, Publ. Mat., **48** (2004), pp. 3–48.
14. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.
15. F. Nazarov, Y. Peres, A. Volberg *The power law for the Buffon needle probability of the four-corner Cantor set*, arXiv:0801.2942, 2008, pp. 1–15.
16. Y. Peres, K. Simon and B. Solomyak, *Self-similar sets of zero Hausdorff measure and positive packing measure*, *Israel J. Math.* **117** (2000),353–379.
17. Y. Peres and B. Solomyak, *How likely is Buffon's needle to fall near a planar Cantor set? Pacific J. Math. 204*, 2 (2002), 473–496.
18. I. J. Schoenberg, *On the Besicovitch-Perron solution of the Kakeya problem*, *Studies in mathmatical analysis and related topics*,
19. T. Tao, *A quantitative version of the Besicovitch projection theorem via multiscale analysis,* pp. 1–28, arXiv:0706.2446v1 [math.CA] 18 Jun 2007.