A_1 CONJECTURE: WEAK NORM ESTIMATES OF WEIGHTED SINGULAR OPERATORS AND BELLMAN FUNCTIONS

FEDOR NAZAROV, ALEXANDER REZNIKOV, VASILY VASYUNIN, AND ALEXANDER VOLBERG

Abstract. We consider several weak type estimates for dyadic singular operators using the Bellman function approach. We disprove the A_1 conjecture, which is a weaker statement than Muckenhoupt–Wheeden conjecture.

1. Introduction

Recently Maria Reguera [4] disproved Muckenhoupt–Wheeden conjecture. Here we strengthen her result by disproving a weaker conjecture: the so called A_1 conjecture. The reader can get acquainted with the best so far positive result on A_1 conjecture in the paper [1]. Then Maria Reguera and Christoph Thiele disproved Muckenhoupt–Wheeden conjecture [5], which required that the Hilbert transform would map $L^1(Mw)$ into $L^{1,\infty}(w)$. It has been suggested in Pérez' paper [3] that there should exist such a counterexample, also [3] has several very interesting positive results, where Mw is replaced by a slightly bigger maximal function, in particular by M^2w (which is equivalent to a certain Orlicz maximal function).

We disprove a weaker conjecture that required that the Hilbert transform would map $L^1(w)$ to $L^{1,\infty}(w)$ with norm bounded by constant times $[w]_{A_1}$–the A_1 "norm" of w. Recall that $[w]_{A_1} := \sup \frac{Mw(x)}{w(x)}$. Therefore, A_1 conjecture is weaker than Muckenhoupt –Wheeden conjecture. And, in fact, in [4], [5] the A_1 norm of the weight is uncontrolled.

It is called a weak Muckenhoupt–Wheeden conjecture. We prove that the linear estimate in weak Muckenhoupt –Wheeden conjecture is impossible, and, moreover, the growth of the norm of the the Hilbert transform from $L^1(w)$ into $L^{1,\infty}(w)$ is at least $c\,[w]_{A_1}\log^{1/7}[w]_{A_1}$. Paper [1] gives an estimate from above for such a norm: it is $\leq C\,[w]_{A_1}\log[w]_{A_1}$. We believe that this might be sharp and that our estimates from below can be improved.

Acknowledgements. This work is much indebted to the Summer School-2010 at Lake Arrowhead and the discussions during this very stimulating event.

2. Unweighted weak type of 0 shift

We are on $I_0 := [0,1]$. As always D denote the dyadic lattice. We consider the operator

$$\varphi \to \sum_{I\subseteq I_0, I\in D} \epsilon_I(\varphi, h_I)h_I\,,$$

where $\epsilon_I = \pm 1$. Notice that the sum does not contain the constant term.

Put

$$F := \langle|\varphi|\rangle_I\,,\; f := \langle\varphi\rangle_I\,,$$

1991 *Mathematics Subject Classification.* 30E20, 47B37, 47B40, 30D55.

Key words and phrases. Key words: Calderón–Zygmund operators, A_2 weights, A_1 weights, Carleson embedding theorem, Corona decomposition, stopping time, nonhomogeneous Harmonic Analysis, extrapolation, weak type .

and introduce the following function:

$$B(F,f,\lambda) := \sup \frac{1}{|I|}|\{x\in I : \sum_{J\subseteq I, J\in D} \epsilon_J(\varphi,h_J)h_J(x) > \lambda\}|\,,$$

where the *sup* is taken over all ϵ_J, $J\in D$, $J\subseteq I$, and over all $\varphi\in L^1(I)$ such that $F := \langle|\varphi|\rangle_I$, $f := \langle\varphi\rangle_I$, h_I are normalized in $L^2(\mathbb{R})$ Haar function of the cube (interval) I, and $|\cdot|$ denote Lebesgue measure. Recall that

$$h_I(x) := \begin{cases} \frac{1}{\sqrt{|I|}}\,, \ x\in I_+ \\ -\frac{1}{\sqrt{|I|}}\,, \ x\in I_- \end{cases}$$

This function is defined in a convex domain $\Omega\subset\mathbb{R}^3$: $\Omega := \{(F,f,\lambda)\in\mathbb{R}^3 : |f|\le F\}$.
Remark. Function B should not be indexed by I because it does not depend on I. We will use this soon.

2.1. **Main inequality.**

Theorem 2.1. *Let $P, P_+, P_- \in\Omega, P = (F,f,\lambda)$, $P_+ = (F+\alpha, f+\beta, \lambda+\beta)$, $P_- = (F-\alpha, f-\beta, \lambda-\beta)$. Then*

$$B(P) - \frac{1}{2}(B(P_+) + B(P_-)) \ge 0\,. \tag{2.1}$$

At the same time, if $P, P_+, P_- \in\Omega, P = (F,f,\lambda)$, $P_+ = (F+\alpha, f+\beta, \lambda-\beta)$, $P_- = (F-\alpha, f-\beta, \lambda+\beta)$. Then

$$B(P) - \frac{1}{2}(B(P_+) + B(P_-)) \ge 0\,. \tag{2.2}$$

In particular, this means that denoting $M(F,y_1,y_2) := B(F, y_1+y_2, y_1-y_2)$ defined in domain $G := \{(F,y_1,y_2) : |y_1+y_2|\le F\}$, we get that for each fixed y_2, $M(F,y_1,\cdot)$ is concave and for each fixed y_1, $M(F,\cdot,y_2)$ is concave.

Proof. Fix $P, P_+, P_- \in\Omega, P = (F,f,\lambda)$, $P_+ = (F+\alpha, f+\beta, \lambda+\beta)$, $P_- = (F-\alpha, f-\beta, \lambda-\beta)$. Let φ_+,φ_- be functions giving the supremum in $B(P_+), B(P_-)$ respectively up to a small number $\eta>0$. Using the remark above we think that φ_+ is on I_+ and φ_- is on I_-. Consider

$$\varphi(x) := \begin{cases} \varphi_+(x)\,, \ x\in I_+ \\ \varphi_-(x)\,, \ x\in I_- \end{cases}$$

Notice that then

$$(\varphi, h_I)\cdot\frac{1}{\sqrt{|I|}} = \beta\,. \tag{2.3}$$

Then it is easy to see that

$$\langle|\varphi|\rangle_I = F = P_1,\ \ \langle\varphi\rangle_I = f = P_2\,. \tag{2.4}$$

Notice that for $x\in I_+$ using (2.3), we get if $\epsilon_I = -1$

$$\frac{1}{|I|}|\{x\in I_+ : \sum_{J\subseteq I, J\in D}\epsilon_J(\varphi,h_J)h_J(x) > \lambda\}| = \frac{1}{|I|}|\{x\in I_+ : \sum_{J\subseteq I_+, J\in D}\epsilon_J(\varphi,h_J)h_J(x) > \lambda+\beta\}|$$

$$= \frac{1}{2|I_+|}|\{x\in I_+ : \sum_{J\subseteq I_+, J\in D}\epsilon_J(\varphi_+,h_J)h_J(x) > P_{+,3}\}| \ge \frac{1}{2}B(P_+) - \eta\,.$$

Similarly, for $x \in I_-$ using (2.3), we get if $\epsilon_I = -1$

$$\frac{1}{|I|}|\{x \in I_- : \sum_{J\subseteq I, J\in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda\}| = \frac{1}{|I|}|\{x \in I_- : \sum_{J\subseteq I_+, J\in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda - \beta\}|$$

$$= \frac{1}{2|I_-|}|\{x \in I_- : \sum_{J\subseteq I_-, J\in D} \epsilon_J(\varphi_-, h_J)h_J(x) > P_{-,3}\}| \geq \frac{1}{2}B(P_-) - \eta\,.$$

Combining the two left hand sides we obtain for $\epsilon_I = -1$

$$\frac{1}{|I|}|\{x \in I_+ : \sum_{J\subseteq I, J\in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda\}| \geq \frac{1}{2}(B(P_+) + B(P_-)) - 2\eta\,.$$

Let us use now the simple information (2.4): if we take the supremum in the left hand side over all functions φ, such that $\langle|\varphi|\rangle_I = F, \langle\varphi\rangle_I = f$, and supremum over all $\epsilon_J = \pm 1$ (only $\epsilon_I = -1$ stays fixed), we get a quantity smaller or equal than the one, where we have the supremum over all functions φ, such that $\langle|\varphi|\rangle = F, \langle\varphi\rangle_I = f$, and an unrestricted supremum over all $\epsilon_J = \pm 1$ including $\epsilon_I = \pm 1$. The latter quantity is of course $B(F, f, \lambda)$. So we proved (2.1).

To prove (2.2) we repeat verbatim the same reasoning, only keeping now $\epsilon_I = 1$. We are done.

□

Denote

$$T\varphi := \sum_{J\subseteq I, J\in D} \epsilon_J(\varphi, h_J)h_J(x)\,.$$

It is a dyadic singular operator (actually, it is a family of operators enumerated by sequences of ± 1). To prove tha it is of weak type is the same as to prove

$$B(F, f, \lambda) \leq \frac{C\,F}{\lambda},\ \lambda > 0\,. \tag{2.5}$$

Our B satisfies (2.1), (2.2). We consider this as concavity conditions. And they are if we change variables from (F, f, λ) to (F, y_1, y_2), B to M, where

$$y_1 := \frac{1}{2}(\lambda + f)\,,\ y_2 := \frac{1}{2}(\lambda - f)\,.$$

Abusing the language we will call by the same letter B (correspondingly, M) *any* function satisfying (2.1), (2.2).

We need one more condition, the so-called *obstacle condition*:

(2.6)

If $\lambda \leq F$ then $B(F, f, \lambda) = 1$ or, in terms of M we have $M(F, y_1, y_2) = 1$ if $y_1 + y_2 \leq F$.

Theorem 2.2. *Let $B \geq 0$ satisfy* (2.1), (2.2). *(Equivalently, let the corresponding $M \geq 0$ be concave in (F, y_1) and in (F, y_2).) Let B satisfy* (2.5)*, or, equivalently,*

$$M(F, y_1, y_2) \leq \frac{C\,F}{y_1 + y_2},\ y_1 + y_2 > 0\,. \tag{2.7}$$

Let B (or M) satisfy (2.6)*. Then we have the weak type estimate with constant at most C for all T uniformly in $\epsilon_I = \pm 1$.*

Proof. Just by reversing the argument of Theorem 2.1. OR BEAUTIFULLY BY INDUCTION!

□

3. Unweighted case. The exact Bellman function

Consider finding the unweighted Bellman function on the boundary $F = f$. It means that we are working with positive functions φ. If we denote as before

$$y_1 = \frac{1}{2}(\lambda + f)\,,\ y_2 = \frac{1}{2}(f - \lambda)$$

then we need 0-homogeneous function in y_1, y_2, that is a function $B(\frac{y_1}{y_2})$, such that $B(t)$ is defined for $t \in (-\infty, -1]$ such that

$$\left(B(\frac{y_1}{y_2})\right)''_{y_1y_1} \le 0\,,\ \left(B(\frac{y_1}{y_2})\right)''_{y_2y_2} \le 0 \tag{3.1}$$

and knowing that when $\lambda \to \infty$ we have $\frac{y_1}{y_2} \to -1$, we obtain that

$$B(-1) = 0\,. \tag{3.2}$$

On the other hand, we want to show that

$$B(t) \to 1,\, t \to -\infty\,. \tag{3.3}$$

To prove (3.3) consider first a positive function φ with $\int_0^1 \varphi\, dx = 1$, which looks like almost a delta function. Then $\varphi = 1 - H$, where H is the linear combination of Haar functions which is equal to 1 on almost the whole interval $[0, 1]$, say on a subset E. Then let us consider $\varphi_1 = 1+\epsilon+H$. It corresponds to point $(f_0, \lambda_0) = (1+\epsilon, 1)$, for which $t = \frac{y_1}{y_2} = -\frac{\lambda+f}{\lambda-f} = -\frac{2+\epsilon}{\epsilon}$. But we know that $H = 1 \ge \lambda_0$ on E, $|E| \ge 1-\delta$. We conclude that $B(-\frac{2+\epsilon}{\epsilon}) \ge 1-\delta$, therefore (3.3) is proved.

From (3.1) we derive

$$B'' \le 0,\ t^2B''(t) + 2tB'(t) \le 0,\ \ t \in (-\infty, -1]\,.$$

Make the second inequality an equation. Then we have its concave solution

$$B(t) = c_1 + \frac{c_2}{t}, c_2 > 0\,,$$

the last assumption follows from concavity requirement on B. Now (3.2) and (3.3) give us our desired biconcave function:

$$B(t) = 1 + \frac{1}{t} \Rightarrow L(y_1, y_2) = 1 + \frac{y_2}{y_1} = 1 - \frac{\lambda - f}{\lambda + f} = \frac{2f}{\lambda + f}\,.$$

4. Full Bellman function for unweighted weak type inequality

Amazingly one can also find the full Bellman function, not just on the boundary $f = F$:

$$B(F, f, \lambda) = \begin{cases} 1, \text{ if } \lambda \le F\,, \\ 1 - \frac{(\lambda-F)^2}{\lambda^2-f^2} \text{ if } \lambda > F\,. \end{cases} \tag{4.1}$$

As always above we denote

$$\lambda = y_1 + y_2\,,\ f = y_1 - y_2\,,\ y_1 = \frac{1}{2}(\lambda + f)\,,\ y_2\frac{1}{2}(\lambda - f)\,. \tag{4.2}$$

We are looking for a function M

$$B(F, f, \lambda) = B(F, y_1 - y_2, y_1 + y_2) =: M(F, y_1, y_2)\,,$$

which is defined in $\Omega := \{(F, y_1, y_2) : F \geq |y_1 - y_2|\}$, bi-concave in variables (F, y_1) and (F, y_2), satisfies the first boundary condition:

$$\text{If } F = f := y_1 - y_2 \geq 0 \Rightarrow B(F, f, \lambda) := \frac{2f}{\lambda + f} \Rightarrow M(F, y_1, y_2) = 1 - \frac{y_2}{y_1}\,. \tag{4.3}$$

This we know from Section 3.

It also satisfies the second boundary condition:

$$\text{If } \lambda \leq f \text{ that is if } y_2 \leq 0 \Rightarrow M = 1\,. \tag{4.4}$$

In fact, fix λ_0 and $\epsilon > 0$, let φ be a positive function on $[0, 1]$ such that it looks like $(\lambda_0+\epsilon)\delta_0$, and $F = f := \int_0^1 \varphi\, dx = \lambda_0 + \epsilon > \lambda_0$. As it looks as a multiple of delta function it can be written down as $\lambda_0 + \epsilon + H$, where H is a combination of Haar functions, and $-H = \lambda_0 + \epsilon$ on a set of measure $1 - \tau$ with an arbitrary small τ (the smallness is independent of λ_0 and ϵ). Then the example of φ shows that

$$B(\lambda_0 + \epsilon, \lambda_0 + \epsilon, \lambda_0) \geq 1 - \tau\,.$$

Now let us modify φ a bit. Consider $\Phi := \varphi + S$, where S has very small support (say, of measure smaller than τ), contained in where φ is small, and $\int S\, dx = 0$, and $\int |S|\, dx = N$, which we choose as large as we want. Then the example of φ shows that

$$B(\int_0^1 |\Phi|\, dx, \lambda_0 + \epsilon, \lambda_0) \geq 1 - 2\tau\,.$$

By varying S we can reach $\int |\Phi|\, dx = F$ for any $F \geq \lambda_0 + \epsilon$. Therefore, making first $\tau \to 0$ and then $\epsilon \to 0$, we prove (4.4).

Now let us consider the section Ω_{y_1} of the domain of definition $\Omega := \{(F, y_1, y_2) : F \geq |y_1 - y_2|\}$ by the hyperplane $y_1 = \mathit{fixed}$. We want to find M satisfying concavity in this hyperplane–we are going to look for M (and we will check later that it is concave) that solves Monge–Ampère (MA) equation in Ω_{y_1} with boundary conditions (4.3) and (4.4). In Ω_{y_1}, there is a point $P := (y_1, y_1, 0)$. Let us make a guess that the characteristics (and we know by Pogorelov's theorem that they form the foliation of Ω_{y_1} by straight lines) of our MA equation in Ω_{y_1} form the fan of lines with common point $P = (y_1, y_1, 0)$. By Pogorelov's theorem we also know that there exists functions t_1, t_2, t constant on characteristics such that

$$M = t_1 F + t_2 y_2 + t\,, \tag{4.5}$$

such that $t_1 = t_1(t; y_1), t_2 = t_2(t; y_1)$ (we think that y_1 is a parameter), that

$$0 = (t_1)'_t F + (t_2)'_t y_2 + 1\,, \tag{4.6}$$

that

$$t_1 = \frac{\partial M(F, \cdot, y_2)}{\partial F}\,,\; t_1 = \frac{\partial M(F, \cdot, y_2)}{\partial y_2}\,. \tag{4.7}$$

Let us call characteristics L_t. Extend one of them from P till $y_2 = y_1$. The latter is the vertical line in Ω_{y_1}, which is the intersection of Ω_{y_1} with $f = 0$. Our function B is even with respect to f, being (as we will see this will be true) smooth it has one more boundary condition: $\frac{\partial B}{\partial f}(F, 0, \lambda) = 0$, that is

$$\text{If } y_2 = y_1 \Rightarrow \frac{\partial M}{\partial y_2} = \frac{\partial M}{\partial y_1}\,. \tag{4.8}$$

Or if we denote the intersection of L_t with $y_2 = y_1$ by $(F(t), y_1, y_1)$ we get

$$t_2(t;y_1)=\frac{\partial M}{\partial y_1}(F(t),y_1,y_1)\,. \tag{4.9}$$

We want to prove now that

$$\text{On the whole } L_t \text{ we have } F(t)t_1+2y_1t_2=0\,. \tag{4.10}$$

In fact, our M is 0 homogeneous. So everywhere $FM'_F+y_1M'_{y_1}+y_2M'_{y_2}=0$. Apply this to point $(F(t),y_1,y_1)$, where we can use (4.9) and get $F(t)t_1+t_2y_1+t_2y_1=0$, which is (4.10) in one point. But then all entries are constants on L_t, therefore, (4.10) follows.

Now use our guess that L_t fan from $P=(y_1,y_1,0)$. Plug this coordinates into $0=(t_1)'_tF+(t_2)'_ty_2+1$, which is (4.6). Then we get the crucial (and trivial) ODE

$$t_1'(t)=-\frac{1}{y_1}\Rightarrow t_1(t)=-\frac{1}{y_1}t+C_1(y_1)\,. \tag{4.11}$$

Let boundary line $F=y_1-u$ corresponds to $t=t_0$. Then we use (4.5) and (4.3):

$$(-\frac{1}{y_1}t_0+C_1(y_1))(y_1-u)+t_2u+t_0=1-\frac{u}{y_1}\,.$$

Using (4.10) we can plug t_2 expressed via $F(t)$. But by definition $F(t_0)=0$. So we get

$$(-\frac{1}{y_1}t_0+C_1(y_1))(y_1-u)+t_0=1-\frac{u}{y_1}\,.$$

Or

$$C_1(y_1)y_1-(t_0+C_1(y_1)y_1)\frac{u}{y_1}=1-\frac{u}{y_1}\,.$$

Varying u we get $C_1(y_1)=\frac{1}{y_1}$, $t_0=0$. Now from (4.11) we get

$$t_1(t)=\frac{1}{y_1}(1-t)\,. \tag{4.12}$$

After that (4.6) and (4.10) become the system of two linear "ODE"s on $F(t)$ and $t_2(t)$:

$$\begin{cases}-\frac{1}{y_1}F(t)+y_1t_2'(t)+1=0\\ 2y_1t_2(t)+F(t)\frac{1}{y_1}(1-t)=0\,.\end{cases} \tag{4.13}$$

We find $t_2=-\frac{1}{y_1}(1-t)t$. We find the arbitrary constant for t_2 by noticing that the second equation of (4.13) at $t_0=0$ implies that $t_2(0)=0$ as $F(t_0)=F(0)=0$ by definition.

Hence (4.6) becomes

$$-\frac{1}{y_1}F+\frac{1}{y_1}(2t-1)y_2+1=0\,. \tag{4.14}$$

Given $(F,y_1,y_2)\in\Omega_{y_1}\cap\{0\le y_2\le y_1\}$, we find t from (4.14) and plug it into (4.5), in which we know already $t_(t)$ and $t_2(t)$. Namely, we know that

$$M(F,y_1,y_2)=\frac{1}{y_1}F-\frac{1}{y_1}t(1-t)y_2+t\,. \tag{4.15}$$

Plugging $t = \frac{1}{2}\frac{F-(y_1-y_2)}{y_2}$ (notice that in (F, F, λ) coordinates we will get $t = \frac{F-f}{\lambda-f}$) from (4.14) into this equation we finally obtain

$$B(F, f, \lambda) = 1 - \frac{(\lambda - F)^2}{\lambda^2 - f^2}\,. \tag{4.16}$$

It is not difficult to check now that $M(F, y_1, y_2) = B(F, y_1 - y_2, y_1 + y_2)$ is bi-concave and satisfies all boundary conditions. So we verified that actually that this is exact Bellman function of unweighted weak type estimate.

5. Back to weighted estimate. A_1 case

We keep the notations–almost. Now w will be not an arbitrary weight but a dyadic A_1 weight. Meaning that

$$\forall I \in D\ \langle w\rangle_I \le Q \inf_I w\,.$$

The best Q is called $[w]_{A_1}$. Now

$$F = \langle |f| w\rangle_I, f = \langle f, \rangle_I, \lambda = \lambda, w = \langle w\rangle_I, m = \inf_I w\,.$$

We are in the domain

$$\Omega := \{(F, w, m, f, \lambda) : F \ge |f|\, m,\ \ m \le w \le Q\, w\}\,. \tag{5.1}$$

Introduce

$$\mathbb{B}(F, w, m, f, \lambda) := \sup \frac{1}{|I|} w\{x \in I : \sum_{J \subseteq I, J \in D} \epsilon_J(\varphi, h_J) h_J(x) > \lambda\}\,, \tag{5.2}$$

where the *sup* is taken over all $\epsilon_J, |\epsilon_J| \le 1, J \in D,\ J \subseteq I$, and over all $f \in L^1(I, wdx)$ such that $F := \langle |f|\, w\rangle_I\,,\ f := \langle f\rangle_I,\ w = \langle w\rangle_I, m \le \inf_I w$, and w are dyadic A_1 weights, such that $\forall I \in D\ \langle w\rangle_I \le Q \inf_I w$, and Q being the best such constant. In other words $Q := [w]_{A_1}^{dyadic}$. Recall that h_I are normalized in $L^2(\mathbb{R})$ Haar function of the cube (interval) I, and $|\cdot|$ denote Lebesgue measure.

5.1. **Homogeneity.** By definition, it is clear that

$$s\mathbb{B}(F/s, w/s, m/s, f, \lambda) = \mathbb{B}(F, w, m, f, \lambda)\,,$$

$$\mathbb{B}(tF, w, m, tf, t\lambda) = \mathbb{B}(F, w, m, f, \lambda)\,.$$

Choosing $s = m$ and $t = \lambda^{-1}$, we can see that

$$\mathbb{B}(F, w, m, f, \lambda) = mB(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}) \tag{5.3}$$

for a certain function B. Introducing new variables $\alpha = \frac{F}{m\lambda}, \beta = \frac{w}{m}, \gamma = \frac{f}{\lambda}$ we write that B is defined in

$$G := \{(\alpha, \beta, \gamma) : |\gamma| \le \alpha, 1 \le \beta \le Q\}\,. \tag{5.4}$$

5.2. **The main inequality.**

Theorem 5.1. *Let* $P, P_+, P_- \in \Omega, P = (F, w, \min(m_+, m_-), f, \lambda)$, $P_+ = (F+\alpha, w+\gamma, m_+, f+\beta, \lambda+\beta)$, $P_- = (F-\alpha, w-\gamma, m_-, f-\beta, \lambda-\beta)$. *Then*

$$\mathbb{B}(P) - \frac{1}{2}(\mathbb{B}(P_+) + \mathbb{B}(P_-)) \geq 0\,. \tag{5.5}$$

At the same time, if $P, P_+, P_- \in \Omega, P = (F, f, w, \min(m_+, m_-), \lambda)$, $P_+ = (F+\alpha, f+\beta, w+\gamma, m_+, \lambda-\beta)$, $P_- = (F-\alpha, f-\beta, w+\gamma, m_+, \lambda+\beta)$. *Then*

$$\mathbb{B}(P) - \frac{1}{2}(\mathbb{B}(P_+) + \mathbb{B}(P_-)) \geq 0\,. \tag{5.6}$$

In particular, with fixed m, *and with all points being inside* Ω *we get*

$$\mathbb{B}(F, w, m, f, \lambda) - \frac{1}{4}(\mathbb{B}(F-dF, w-dw, m, f-d\lambda, \lambda-d\lambda) + \mathbb{B}(F-dF, w-dw, m, f+d\lambda, \lambda-d\lambda) +$$

$$\mathbb{B}(F+dF, w+dw, m, f-d\lambda, \lambda+d\lambda) + \mathbb{B}(F+dF, w+dw, m, f+d\lambda, \lambda+d\lambda)) \geq 0\,. \tag{5.7}$$

Remark.1) Differential notations $dF, dw, d\lambda$ just mean small numbers. 2) In (5.7) we loose a bit of information (in comparison to (2.1),(2.2)), but this is exactly (5.7) that we are going to use in the future.

Proof. Fix $P, P_+, P_- \in \Omega$. Let φ_+, φ_-, w_+, w_- be functions and weights giving the supremum in $B(P_+), B(P_-)$ respectively up to a small number $\eta > 0$. Using the fact that $\mathbb{B}$ does not depend on I, we think that φ_+, w_+ is on I_+ and φ_-, w_- is on I_-. Consider

$$\varphi(x) := \begin{cases} \varphi_+(x)\,, \ x \in I_+ \\ \varphi_-(x)\,, \ x \in I_- \end{cases}$$

$$\omega(x) := \begin{cases} w_+(x)\,, \ x \in I_+ \\ w_-(x)\,, \ x \in I_- \end{cases}$$

Notice that then

$$(\varphi, h_I) \cdot \frac{1}{\sqrt{|I|}} = \beta\,. \tag{5.8}$$

Then it is easy to see that

$$\langle |\varphi|\omega \rangle_I = F = P_1, \ \ \langle \varphi \rangle_I = f = P_4\,. \tag{5.9}$$

Notice that for $x \in I_+$ using (5.8), we get if $\epsilon_I = -1$

$$\frac{1}{|I|} w_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \epsilon_J (\varphi, h_J) h_J(x) > \lambda\} = \frac{1}{|I|} w_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \epsilon_J (\varphi, h_J) h_J(x) > \lambda + \beta\}$$

$$= \frac{1}{2|I_+|} w_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \epsilon_J (\varphi_+, h_J) h_J(x) > P_{+,3}\} \geq \frac{1}{2} B(P_+) - \eta\,.$$

Similarly, for $x \in I_-$ using (5.8), we get if $\epsilon_I = -1$

$$\frac{1}{|I|} w_-\{x \in I_- : \sum_{J \subseteq I, J \in D} \epsilon_J (\varphi, h_J) h_J(x) > \lambda\} = \frac{1}{|I|} w_-\{x \in I_- : \sum_{J \subseteq I_-, J \in D} \epsilon_J (\varphi, h_J) h_J(x) > \lambda - \beta\}$$

$$= \frac{1}{2|I_-|} w_-\{x \in I_- : \sum_{J\subseteq I_-, J\in D} \epsilon_J(\varphi_-, h_J)h_J(x) > P_{-,3}\} \geq \frac{1}{2}B(P_-) - \eta\,.$$

Combining the two left hand sides we obtain for $\epsilon_I = -1$

$$\frac{1}{|I|}\omega\{x \in I_+ : \sum_{J\subseteq I, J\in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda\} \geq \frac{1}{2}(B(P_+) + B(P_-)) - 2\eta\,.$$

Let us use now the simple information (5.9): if we take the supremum in the left hand side over all functions φ, such that $\langle|\varphi|\, w\rangle_I = F, \langle\varphi\rangle_I = f, \langle\omega\rangle = w$, and weights ω: $\langle\omega\rangle = w$, in dyadic A_1 with A_1-norm at most Q, and supremum over all $\epsilon_J = \pm 1$ (only $\epsilon_I = -1$ stays fixed), we get a quantity smaller or equal than the one, where we have the supremum over all functions φ, such that $\langle|\varphi|\, \omega\rangle = F, \langle\varphi\rangle_I = f, \langle\omega\rangle = w$, and weights ω: $\langle\omega\rangle = w$, in dyadic A_1 with A_1-norm at most Q, and an unrestricted supremum over all $\epsilon_J = \pm 1$ including $\epsilon_I = \pm 1$. The latter quantity is of course $\mathbb{B}(F, w, m, f, \lambda)$. So we proved (2.1).

To prove (2.2) we repeat verbatim the same reasoning, only keeping now $\epsilon_I = 1$. We are done.

□

Remark. This theorem is a sort of "fancy" concavity property, the attentive reader would see that (5.5), (5.6) represent bi-concavity not unlike demonstrated by the celebrated Burkholder's function. We will use the consequence of bi-concavity encompassed by (5.7). There is still another concavity if we allow to have $|\epsilon_J| \leq 1$.

Theorem 5.2. *In the definition of $\mathbb{B}$ we allow now to take supremum over all $|\epsilon_j| \leq 1$. Let $P, P_+, P_- \in \Omega, P = (F, w, m, f, \lambda)$, $P_+ = (F + \alpha, w + \gamma, m, f + \beta, \lambda)$, $P_- = (F - \alpha, w - \gamma, m, f - \beta, \lambda)$. Then*

$$\mathbb{B}(P) - \frac{1}{2}(\mathbb{B}(P_+) + \mathbb{B}(P_-)) \geq 0\,. \tag{5.10}$$

Proof. We repeat the proof of (5.5) but with $\epsilon_I = 0$.

□

Theorem 5.3. *For fixed F, w, f, λ function $\mathbb{B}$ is decreasing in m.*

Proof. Let $m = \min(m_-, m_=) = m_-$. And let $m_+ > m$. Then (5.5) becomes

$$\mathbb{B}(F, w, m, f, \lambda) - \mathbb{B}(F, w, m_+, f, \lambda) \geq 0\,.$$

This is what we want. □

5.3. **Differential properties of $\mathbb{B}$ translated to differential properties of B.** It is convenient to introducae an auxillairy function of 4 variables:

$$\widetilde{B}(x, y, f, \lambda) := B(\frac{x}{\lambda}, y, \frac{f}{\lambda})\,.$$

Of course

$$\mathbb{B}(F, w, m, f, \lambda) = m\widetilde{B}(\frac{F}{m}, \frac{w}{m}, f, \lambda)\,.$$

Theorem 5.4. *Funcion B from (5.3) satisfies*

$$t \to t^{-1}B(\alpha t, \beta t, \gamma) \textit{ is increasing for } \frac{|\gamma|}{\alpha} \leq t \leq \frac{Q}{\beta}\,. \tag{5.11}$$

$$B \textit{ is concave}\,. \tag{5.12}$$

$$B(\frac{x}{\lambda},y,\frac{f}{\lambda})-\frac{1}{4}\Big[B(\frac{x-dx}{\lambda-d\lambda},y-dy,\frac{f-d\lambda}{\lambda-d\lambda})+B(\frac{x-dx}{\lambda-d\lambda},y-dy,\frac{f+d\lambda}{\lambda-d\lambda})+$$

$$B(\frac{x+dx}{\lambda+d\lambda},y-dy,\frac{f-d\lambda}{\lambda+d\lambda})+B(\frac{x+dx}{\lambda+d\lambda},y-dy,\frac{f+d\lambda}{\lambda-d\lambda}\Big]\geq 0\,. \tag{5.13}$$

Proof. These relations follow from Theorem 5.3, Theorem 5.2, and Theorem 5.1 (actually from (5.7)) correspondingly. □

We can choose extremely small ε_0 and inside the domain Ω we can mollify $\mathbb{B}$ by multiplicative convolution of it with ε_0-bell function ψ supported in a ball of radius $\varepsilon_0/2$.

Multiplicative convolution can be viewed as the integration with $\frac{1}{\delta^5}\psi(\frac{x-x_0}{\delta})$, where $\delta=\mathrm{dist}(x_0,\partial\Omega)$. Here x_0 is a point inside the domain of definition Ω for function $\mathbb{B}$. This new function we call $\mathbb{B}$ again. We build B by this new $\mathbb{B}$. Actually the new B should be denoted B^{ϵ_0}, where superscript denotes our operation of mollification. Property (5.13) can be now rewritten for it by use of Taylor's formula:

Theorem 5.5.

$$-\alpha^2B_{\alpha\alpha}\left(\frac{dx}{x}-\frac{d\lambda}{\lambda}\right)^2-\beta^2B_{\beta\beta}\left(\frac{dy}{y}\right)^2-(1+\gamma^2)B_{\gamma\gamma}\left(\frac{d\lambda}{\lambda}\right)^2-$$
$$-2\alpha\beta B_{\alpha\beta}\left(\frac{dx}{x}-\frac{d\lambda}{\lambda}\right)\frac{dy}{y}+2\beta\gamma B_{\beta\gamma}\frac{dy}{y}\frac{d\lambda}{\lambda}+2\alpha\gamma B_{\alpha\gamma}\left(\frac{dx}{x}-\frac{d\lambda}{\lambda}\right)\frac{d\lambda}{\lambda}+$$
$$+2\alpha B_\alpha\left(\frac{dx}{x}-\frac{d\lambda}{\lambda}\right)\frac{d\lambda}{\lambda}-2\gamma B_\gamma\left(\frac{d\lambda}{\lambda}\right)^2\geq 0\,.$$

Proof. This is just Taylor's formula applied to (5.13). □

Denoting

$$\xi=\frac{dx}{x}=\frac{dy}{y}\,,\ \eta=\frac{d\lambda}{\lambda}$$

we obtain the following quadratic form inequality

Theorem 5.6.

$$-\xi^2\left[\alpha^2B_{\alpha\alpha}+\beta^2B_{\beta\beta}+2\alpha\beta B_{\alpha\beta}\right]-\eta^2\left[\alpha^2B_{\alpha\alpha}+(1+\gamma^2)B_{\gamma\gamma}+2\alpha\gamma B_{\alpha\gamma}2\alpha B_\alpha+2\gamma B_\gamma\right]+$$
$$+2\xi\eta\left[\alpha^2B_{\alpha\alpha}+\alpha\beta B_{\alpha\beta}\beta\gamma B_{\beta\gamma}+\alpha\gamma B_{\alpha\gamma}+\alpha B_\alpha\right]\geq 0\,.$$

Now let us combine Theorem 5.6 and Theorem 5.2. In fact, Theorem 5.2 implies

$$-2\alpha\gamma B_{\alpha\gamma}\eta^2\leq-\alpha^2\gamma^2B_{\alpha\alpha}\eta^2-b_{\gamma\gamma}\eta^2\,.$$

We plug it into the second term above. Also Theorem 5.2 implies

$$2\alpha\gamma B_{\alpha\gamma}\xi\eta\leq-\alpha^2\gamma^2B_{\alpha\alpha}\xi^2-B_{\gamma\gamma}\eta^2\,,$$
$$2\beta\gamma B_{\beta\gamma}\xi\eta\leq-\beta^2\gamma^2B_{\beta\beta}\xi^2-B_{\gamma\gamma}\eta^2\,,$$

We will plug it into the third term above. Then using the notation

$$\psi(\alpha\beta,\gamma):=-\alpha^2B_{\alpha\alpha}-2\alpha\beta B_{\alpha\beta}-\beta^2B_{\beta\beta})$$

(which is non-negative by the concavity of B in its first two variables by the way) we get that the following quadratic form is non-negative:

$$\xi^2\,K+\xi\eta\,L+\eta^2\,N:=$$
$$\xi^2\left[\psi(\alpha,\beta,\gamma)+(-\alpha^2B_{\alpha\alpha}-\beta^2B_{\beta\beta})\gamma^2\right]$$

$$\xi\eta\,[-\psi(\alpha,\beta,\gamma)+(\alpha^2B_\alpha)_\alpha-\beta^2B_{\beta\beta}]$$
$$\eta^2\,[-(4+\gamma^2)B_{\gamma\gamma}-2\gamma B_\gamma-(\alpha^2B_\alpha)_\alpha-\alpha^2B_{\alpha\alpha}\gamma^2]\geq 0\,.$$

Therefore, (hopefully K is positive),

$$N\geq\frac{L^2}{4K}\,. \tag{5.14}$$

Now we will estimate L from below, K from above and as a result we will obtain the estimate of N from below, which will bring us our proof.

But first we need some a priori estimates, and for that we will need to mollify $B=B^{\epsilon_0}$ in variables α,β. Again we make a multiplicative convolution with a bell-type function. Let us explain why we need it. Let

$$\hat{Q}:=\sup_G B/\alpha\,.$$

We want to prove that

$$\hat{Q}/Q\to\infty\,. \tag{5.15}$$

First we need to notice that

$$\int_{1/2}^1\psi(\alpha t,\beta t,\gamma)\,dt\leq C\,(\hat{Q}\gamma+\frac{\hat{Q}}{Q}\alpha)\,. \tag{5.16}$$

In fact, consider $\beta\in[Q/4,Q/2]$, $b(t):=B(\alpha t,\beta t,\gamma)$ on the interval $\frac{|\gamma|}{\alpha}=:t_0\leq t\leq 2$. Let $\ell(t)=b(2)t\leq\hat{Q}t\alpha$. We saw that $b(t)/t$ is increasing. So b is under ℓ, and so

$$\ell(t)-b(t)|(t=(\max(\frac{\gamma}{\alpha},\frac{c}{Q}))\leq\ell(\max(\frac{\gamma}{\alpha},\frac{c}{Q}))\leq C\hat{Q}\alpha\max(\hat{Q}\frac{\gamma}{\alpha},\frac{1}{Q})\leq\hat{Q}\gamma+\frac{\hat{Q}}{Q}\alpha\,,$$

and $\ell(t)-b(t)$ is concave, and the above value is maximum of it. By the same property that $b(t)/t$ is increasing we get that

$$\ell'(2)-b'(2)\leq 0\,.$$

Combining this with Taylor's formula on $[t_0,2]$ we get for $g:=\ell-b$ (g is convex of course):

$$-(2-t_0)g'(2)+\int_{t_0}^2dt\int_t^2g''(s)ds=\text{positive}+\int_{t_0}^2(s-t_0)g''(s)ds\leq\sup g\leq\hat{Q}\gamma+\frac{\hat{Q}}{Q}\alpha\,.$$

This implies (5.16).

Consider now function $a(t):=B(\alpha t,\beta,\gamma)$ We also have the same type of consideration applied to convex function $\hat{Q}-a(t)$ bringing us

$$\int_{1/2}^1-\alpha^2B_{\alpha\alpha}(\alpha t,\beta,\gamma)\,dt\leq C\hat{Q}\alpha\,. \tag{5.17}$$

Similarly,

$$\int_{1/2}^1-\beta^2B_{\beta\beta}(\alpha,\beta t,\gamma)\,dt\leq C\hat{Q}\alpha\,. \tag{5.18}$$

We tacitly used here that $B_\alpha\geq 0, B_\beta\geq 0$, which is not difficult to see.

For the future estimates we want (5.16), (5.17), (5.18) to hold not in average but pointwise. To do that we just convolve B with a bell-like ψ_{ε_0} in two variables α,β for each fixed γ. It is not difficult to see that after that (5.16), (5.17), (5.18) become

$$0 \le \psi(\alpha t, \beta t, \gamma) \le C(\hat{Q}\gamma + \frac{\hat{Q}}{Q}\alpha)\,. \tag{5.19}$$

$$0 \le -\alpha^2 B_{\alpha\alpha}(\alpha t, \beta, \gamma) \le C\hat{Q}\alpha\,. \tag{5.20}$$

$$0 \le -\beta^2 B_{\beta\beta}(\alpha, \beta t, \gamma) \le C\hat{Q}\alpha\,. \tag{5.21}$$

In fact we should write here $\widetilde{B^{\epsilon_0}}$ instead of B to honor our second mollification. But we will keep B.

Recall that (now with this smoothen B):

$$\xi^2\, K + \xi\eta\, L + \eta^2\, N :=$$
$$\xi^2\, [\psi(\alpha,\beta,\gamma) + (-\alpha^2 B_{\alpha\alpha} - \beta^2 B_{\beta\beta})\gamma^2]$$
$$\xi\eta\, [-\psi(\alpha,\beta,\gamma) + (\alpha^2 B_\alpha)_\alpha - \beta^2 B_{\beta\beta}]$$
$$\eta^2\, [-(4+\gamma^2)B_{\gamma\gamma} - 2\gamma B_\gamma - (\alpha^2 B_\alpha)_\alpha - \alpha^2 B_{\alpha\alpha}\gamma^2] \ge 0\,.$$

We will choose soon appropriate $\alpha_0, \alpha_1 \le \frac{1}{100}\alpha_0$ and $\gamma << \alpha_1$. Let us intruduce

$$k := \int_{\alpha_1}^{\alpha_0} K\, d\alpha = \int_{\alpha_1}^{\alpha_0} [\psi(\alpha,\beta,\gamma) + (-\alpha^2 B_{\alpha\alpha} - \beta^2 B_{\beta\beta})\gamma^2]\, d\alpha\,,$$

$$n := \int_{\alpha_1}^{\alpha_0} N\, d\alpha = \int_{\alpha_1}^{\alpha_0} [-(4+\gamma^2)B_{\gamma\gamma} - 2\gamma B_\gamma - (\alpha^2 B_\alpha)_\alpha - \alpha^2 B_{\alpha\alpha}\gamma^2]\, d\alpha\,,$$

$$\ell := \int_{\alpha_1}^{\alpha_0} [-\psi(\alpha,\beta,\gamma) + (\alpha^2 B_\alpha)_\alpha - \beta^2 B_{\beta\beta}]\, d\alpha\,.$$

Estimate of k from above. The integrand of k is obviously positive and ψ term dominates other terms (by (5.19), (5.20), (5.21) and the smallness of γ). Therefore

$$0 \le k \le \hat{Q}\gamma \log\frac{\alpha_0}{\alpha_1} \le C\,(\hat{Q}\gamma\alpha_0 + C\frac{\hat{Q}}{Q}\alpha_0^2)\,,$$

if Q is very large. We choose

$$\alpha_0 = c\,\frac{Q}{\hat{Q}}\,,\ \alpha_1 = \frac{1}{100}\frac{Q}{\hat{Q}}\alpha_0\,. \tag{5.22}$$

Then

$$k \le C\,(Q\gamma + \frac{Q}{\hat{Q}}) \le C\,\frac{Q}{\hat{Q}}(1 + \hat{Q}\gamma)\,. \tag{5.23}$$

Estimate of ℓ from below. Estimating from below we can skip the non-negative term $-\beta^2 B_{\beta\beta}$. Also

$$\int_{\alpha_1}^{\alpha_0} -\psi(\alpha,\beta,\gamma) \ge -CQ\gamma - C\frac{Q}{\hat{Q}}\,.$$

On the other hand,

$$\int_{\alpha_1}^{\alpha_0} (\alpha^2 B_\alpha)_\alpha\, d\alpha = \alpha_0^2 B_\alpha(\alpha_0,\beta,\gamma) - \alpha_1^2\hat{Q}\,,$$

as mollification gives a pointwise estimate

$$B_\alpha \le C\hat{Q}\,. \tag{5.24}$$

Recall that $\beta \in [Q/4, Q/2]$. We also have the obstacle condition (5.30), which says that $B(1,\beta,\gamma) \ge \frac{\beta}{8}$. If $B_\alpha(\alpha_0,\beta,\gamma)$ would be smaller than $Q/40$ (and then $B_\alpha(s,\beta,\gamma) \le Q/40$ for all $s \in [\alpha_0, 1]$ by concavity of B in its first variable) we would not be able to reach at least $\frac{Q}{4\cdot 8}$.

$$\ell \ge \frac{\alpha_0^2}{40} Q - \alpha_1^2 \hat{Q} - CQ\gamma - C\frac{Q}{\hat{Q}}\,.$$

As $\alpha_1 = \frac{1}{100}\alpha_0 \sqrt{\frac{Q}{\hat{Q}}}$ (see (5.22)), and choosing $\gamma \le \tau(\frac{Q}{\hat{Q}})^2$ we get

$$\ell \ge \frac{\alpha_0^2}{80} Q \ge c(\frac{Q}{\hat{Q}})^2 Q\,.$$

And k is

$$0 \le k \le C\,\frac{\hat{Q}}{Q}(1+\hat{Q}\gamma)\,,$$

As we must have

$$n \ge \frac{\ell^2}{4k} \ge C\,(\frac{Q}{\hat{Q}})^5\,\frac{Q^2}{1+\hat{Q}\gamma}\,. \tag{5.25}$$

Estimate of n from above. As $\gamma << c\frac{Q}{\hat{Q}}$, we have that (see (5.20))

$$\int_{\alpha_1}^{\alpha_0} -(\alpha^2 B_\alpha)_\alpha\, d\alpha - \gamma^2 \int_{\alpha_1}^{\alpha_0} -\alpha^2 B_{\alpha\alpha}\, d\alpha \le -CQ\alpha_0^2 + c\hat{Q}\alpha_0^2\gamma^2 \le 0\,.$$

Therefore, we get, combining with (5.25)

$$(\frac{Q}{\hat{Q}})^5\,\frac{Q^2}{1+\hat{Q}\gamma} \le n \le -5\int_{\alpha_1}^{\alpha_0} (e^{\frac{1}{5}\gamma^2} B_\gamma)_\gamma\, d\alpha\,,$$

or

$$\int_{\alpha_1}^{\alpha_0} (e^{\frac{1}{5}\gamma^2} B_\gamma)_\gamma\, d\alpha \le -C(\frac{Q}{\hat{Q}})^7\,\frac{\hat{Q}^2}{1+\hat{Q}\gamma}\,. \tag{5.26}$$

Function B is smooth and symmetric in γ (the latter is by definition). So after integrating in γ on $[0,\gamma], \gamma < \gamma_0 := c\frac{Q}{\hat{Q}}/\log\frac{\hat{Q}}{Q}]$ we get

$$\int_{\alpha_1}^{\alpha_0} B_\gamma\, d\alpha \le -(\frac{Q}{\hat{Q}})^7\,\hat{Q}\log(1+\hat{Q}\gamma)\,. \tag{5.27}$$

Integrate again in γ on $[0,\gamma_0]$. We get the integral over $[\alpha_1,\alpha_0]$ of the oscillation of B, which is

$$\int_{\alpha_1}^{\alpha_0} [B(\alpha,\beta,0) - B(\alpha,\beta,\gamma_0)]\, d\alpha \ge (\frac{Q}{\hat{Q}})^7\,(1+\hat{Q}\gamma_0)\log(1+\hat{Q}\gamma_0)$$

But this oscillation is smaller than $C\hat{Q}\alpha_0^2$. We get the inequality

$$(\frac{Q}{\hat{Q}})^7\,(1+\hat{Q}\gamma_0)\log(1+\hat{Q}\gamma_0) \le (\frac{Q}{\hat{Q}})^2\hat{Q}\,. \tag{5.28}$$

Plugging γ_0 we get

$$(\frac{Q}{\hat{Q}})^5\,(1+\tau\hat{Q}(\frac{Q}{\hat{Q}})^2)\log(1+\tau\hat{Q}(\frac{Q}{\hat{Q}})^2) \le \hat{Q}\,. \tag{5.29}$$

We can see that if $\hat{Q} \le Q \log^\tau Q$ with $\tau < \frac{1}{7}$, then (5.29) leads to a contradiction. So we proved

Theorem 5.7. *The weighted weak norm of the martingale transform for weights $w \in A_1^{dyadic}$ can reach $c\,[w]_{A_1} \log^\tau [w]_{A_1}$ for any positive $\tau < 1/7$.*

5.4. **Obstacle conditions for B.** Now we want to show the following obstacle condition for B, which we already used:

$$\text{if } |\gamma| < \frac{1}{4}\,,\ \text{then } B(1,\beta,\gamma) \ge \frac{\beta}{8}\,. \tag{5.30}$$

Let $I := [0,1]$. Given numbers $|f| < \lambda/4$, $\frac{F}{m} = \lambda$ it is enough to construct functions φ, ψ on I such that

Put $\varphi = -a$ on I_{--}, $= b$ on I_{++}, zero otherwise. And $w = 1$ on $I_{--} \cup I_{++}$, and $w = Q$ otherwise. Then put

$$\psi := (\varphi, h_{I_-})h_{I_-} - (\varphi, h_{I_+})h_{I_+}\,.$$

Let $0 < a < b$ and a is close to b. Put $\lambda = (a+b)/16$. Then average of φ is small with respect to λ and we can prescribe it. $F = (a+b)/4, m = 1$. On the other hand, function ψ (which is a martingale transform of $\varphi - \langle\varphi\rangle$) is at least $-(\varphi, h_{I_+})h_{I_+} \ge \frac{1}{8}b \ge \lambda$ on I_{+-}, whose w-measure is more than $\frac{1}{3}w(I)$. So

$$B(4,\beta,\gamma) \ge \frac{1}{3}\beta\,, \tag{5.31}$$

for all small γ and $\beta \asymp Q$. This is what we wanted to prove. The replacement of 1 by 4 does not matter.

6. Bellman function and the estimate of weighted weak norm from above in A_1 case

Let us denote by N_k the quantity (w is constant on k-th generation and $w \in A_1^{dyadic}$)

$$N_k(V) := \sup \frac{1}{|I|} w\{x \in I : \sum_{J \subset I, J \in D, |J| \ge 2^{-k}|I|} \epsilon_J (f, h_J) h_J > \lambda\}\,.$$

Then we have practically by the definition of N_k (let V temporarily denotes vector (F, f, λ, w, m), and $y_1 := \lambda + f, y_2 := \lambda - f$)

$$N_{k+1}(V) \le \sup_{V_+, V_-, V = \frac{V_+ + V_-}{2}, |y_{1+} - y_{1-}| = |y_{2+} - y_{2-}|} \frac{N_k(V_-) + N_k(V_+)}{2}\,. \tag{6.1}$$

In this language we need to prove that

$$N_k(V) \le B(V) \text{ for any } k \text{ and any } V \in \Omega_k\,. \tag{6.2}$$

By bi-concavity of B and by (6.1) we immediately see the induction step from k to $k+1$. We are left to check that

$$N_0(V) \le B(V)\,. \tag{6.3}$$

Let us check (6.3). If $\lambda > \frac{F}{m} \ge |f|$ we just use $B(V) \ge 0$ because for such parameters

$$|(f, h_{[0,1]})| \le |f| < \lambda$$

and the subset of $[0,1]$, where $\epsilon_{[0,1]}(f, h_{[0,1]}) h_{[0,1]}(x)$ is greater than λ is empty.

On the other hand, if $\lambda \le \frac{F}{m}$, what can be the largest w-measure of $E \subset [0,1]$ on which $\epsilon_{[0,1]}(f, h_{[0,1]}) h_{[0,1]}(x) \ge \lambda$? Here is the extremal situation: w is $2Q-1$ on $[0,1/2]$, and 1 on

$[1/2, 1]$. Function φ is zero on $[0, 1/2]$, and constant $2f$ on $[1/2, 1]$. Then $F = f, m = 1$ (these are data on $[0, 1]$). On the other hand,

$$\epsilon_{[0,1]}(\varphi, h_{[0,1]})h_{[0,1]}(x) = \epsilon_{[0,1]}\, f\, h_{[0,1]}(x) = \frac{F}{m} \geq \lambda$$

on the whole $[0, 1/2]$ if $\epsilon_{[0,1]} = \pm$ is chosen in the right way. But in this case again, $B(V) \geq 2Q - 1 \geq w([0, 1/2])$.

Hence, $B(V) \geq N_0(V)$ is proved, and we can start the induction procedure.

It is left to find our $\mathbb{B}$ to have a sharp estimate from above in A_1 problem.

7. Martingales

We will use fouradic lattice $\mathcal{F}$. For a fouradic interval I let $H_I = 1$ on its right half, $H_I = -1$ on its left half, let also $G_I = 1$ on its leftest and rightest quarters and $G_I = -1$ on two middle quarters. We will call martingale difference the function of the type

$$f_n = \sum_{I \in \mathcal{F}, \ell(I) = 4^{-n}} a_I H_I$$

or

$$g_n = \sum_{I \in \mathcal{F}, \ell(I) = 4^{-n}} b_I G_I\,,$$

where a_I, b_I are numbers. Martingale for us is any function on $I_0 := [0, 1]$ of the type $\mathbf{f} = f + \sum_{n=0}^N f_n$, or $\mathbf{g} = g + \sum_{n=0}^N g_n$, where f, g are two constants. We distinguish H- and G-martingales.

In the previous sections the following theorem was proved.

Theorem 7.1. *Given $Q > 1$ there exist three H-martingales $\mathbf{F}, \mathbf{f}, \mathbf{w}$, $\mathbf{F} \geq \mathbf{0}, \mathbf{w} \geq \mathbf{0}$, and one G-martingale $\mathbf{g}$ such that the following holds:*

1) For any $I \in \mathcal{F}$, $\langle \mathbf{F} \rangle_{\mathbf{I}} \geq \langle |\mathbf{f}| \rangle_{\mathbf{I}} \min_{\mathbf{I}} \mathbf{w}$.

2) For any $I \in \mathcal{F}$, $\langle \mathbf{w} \rangle_{\mathbf{I}} \leq \mathbf{Q} \min_{\mathbf{I}} \mathbf{w}$.

3) For any $I \in \mathcal{F}$, $a_I = b_I$, where these are martingale differences coefficients for $\mathbf{f}$ and $\mathbf{g}$.

4) $\int_{x \in I_0 : \mathbf{g}(x) \leq 0} \mathbf{w}\, \mathbf{dx} \geq \mathbf{c}\, \mathbf{Q} \log^{\mathbf{1/7}} \mathbf{Q} \frac{\mathbf{1}}{\mathbf{g}} \int_{\mathbf{I_0}} \mathbf{F}\, \mathbf{dx}$.

8. Controlled doubling martingales

We are going to make a small modification in the proof to get the following

Theorem 8.1. *Given $Q > 1$ there exist three H-martingales $\mathbf{F}, \mathbf{f}, \mathbf{w}$, $\mathbf{F} \geq \mathbf{0}, \mathbf{w} \geq \mathbf{0}$, and one G-martingale $\mathbf{g}$ such that the following holds:*

1) For any $I \in \mathcal{F}$, $\langle \mathbf{F} \rangle_{\mathbf{I}} \geq \langle |\mathbf{f}| \rangle_{\mathbf{I}} \min_{\mathbf{I}} \mathbf{w}$.

2) For any $I \in \mathcal{F}$, $\langle \mathbf{w} \rangle_{\mathbf{I}} \leq \mathbf{Q} \min_{\mathbf{I}} \mathbf{w}$.

3) For any $I \in \mathcal{F}$, $a_I = b_I$, where these are martingale differences coefficients for $\mathbf{f}$ and $\mathbf{g}$.

4) $\int_{x \in I_0 : \mathbf{g}(x) \leq 0} \mathbf{w}\, \mathbf{dx} \geq \mathbf{c}\, \mathbf{Q} \log^{\mathbf{1/7}} \mathbf{Q} \frac{\mathbf{1}}{\mathbf{g}} \int_{\mathbf{I_0}} \mathbf{F}\, \mathbf{dx}$.

5) For any two fouradic neighbors (neighbors in the tree) $I \in \mathcal{F}$ and $\hat{I}$, $\langle \mathbf{w} \rangle_{\hat{\mathbf{I}}} \leq \mathbf{4} \langle \mathbf{w} \rangle_{\mathbf{I}}$.

In other words, we can always control the fouradic doubling property of $\mathbf{w}$.

9. Remodelling martingale differences

Now we are going to repeat the procedure from [2]. We say that I_0 supervises itself. Take a very large n_1, consider the division of of I_0 to 4^{n_1} small equal intervals and let the leftes quarter of the supervisor (I_0 it is) supervises the first, fifth, etc small subdivision interval of the supervisee (which is still I_0 for now, so these are intervals of our just done subdivision). Let the second quarter supervises the second, the sixth, etc; the third quarter supervises the third, the seventh, etc, and the fourth quarter supervises the fourth, the eighth, etc.

Now we have new pairs of (supervisor, supervisee). Subdivide each supervisor to its 4 sons and its supervisee to 4^{n_2} sons, where $n_2 >> n_1$. Repeat supervisor/supervisee allocation procedure as before, Continue with the new pairs of (supervisor, supervisee). Repeat N times.

Now, as in [2], we are going to "remodel" martingales $\mathbf{w}, \mathbf{F}, \mathbf{f}, \mathbf{g}$ to new functions with basically the same distributions.

We first "square sine" and "square cosine" function for any supervisee interval J. Namely take an absolutely huge M, and let

$$sqsin(x) := H_{I_0}(4^M x)\,,\ sqcos(x) = G_{I_0}(4^M x)\,.$$

Now let l_J be a natural linear map $J \to I_0$. We put $sqs_J := sqsin \circ l_J$, $sqc_J = sqcos \circ l_J$.

Now we basically want to put

$$W := w + \sum_{n=1}^{N} \sum_{I\in\mathcal{F}, \ell(I)=4^{-n}} \sum_{J\, supervised\, by\, I} c_I sqs_J\,,$$

where c_I are coefficients of $\mathbf{w}$.

$$\Phi := F + \sum_{n=1}^{N} \sum_{I\in\mathcal{F}, \ell(I)=4^{-n}} \sum_{J\, supervised\, by\, I} d_I sqs_J\,,$$

where d_I are coefficients of $\mathbf{F}$.

$$\phi := f + \sum_{n=1}^{N} \sum_{I\in\mathcal{F}, \ell(I)=4^{-n}} \sum_{J\, supervised\, by\, I} a_I sqs_J\,,$$

where a_I are coefficients of $\mathbf{f}$.

$$\rho := g + \sum_{n=1}^{N} \sum_{I\in\mathcal{F}, \ell(I)=4^{-n}} \sum_{J\, supervised\, by\, I} b_I sqc_J\,,$$

where $b_I = a_I$ are coefficients of $\mathbf{g}$. Notice that the last formula has square cosines sqc, and this will be important.

We do exactly that, but to ensure the doubling property of W we just for every pair (I, J) (supervisor/supervisee) replace sqs_J by basically the same function, but such that its first 4 steps on the left are replaced by 0 and its first 4 steps on the right are replaced by zero. Call it $sqsm_J$. So

$$W := w + \sum_{n=1}^{N} \sum_{I\in\mathcal{F}, \ell(I)=4^{-n}} \sum_{J\, supervised\, by\, I} c_I sqsm_J\,,$$

where c_I are coefficients of $\mathbf{w}$.

The doubling property of such a W has been checked in [2]. We notice that if $1 << n_1 << n_2 << ... << n_N$ then the distribution functions of these new function are basically the same that for their model martingales. So we can repeat Theorem 8.1. Let us consider the periodic extension of W, Φ, ϕ, ρ to the whole line (or we could consider everything just on the unit circle identifying it with $[0,1)$).

Theorem 9.1. *Given $Q>1$ then the above functions W, Φ, ϕ, ρ are such that the following holds:*

1) For any $J\in\mathcal{F}$, $\langle\Phi\rangle_J \ge \langle|\phi|\rangle_I \min_J W$.
2) For any $J\in\mathcal{F}$, $\langle W\rangle_J \le Q\min_J W$.
3) For any $J\in\mathcal{F}$, $a_J=b_J$, these are martingale differences coefficients for ϕ and ρ.
4) $\int_{x\in I_0:\rho(x)\le 0} W\,dx \ge c_0\,Q\log^{1/7}Q\frac{1}{g}\int_{I_0}\Phi\,dx$.
5) W is doubling with an absolute constant.

Now what happens with the Hilbert transform $H\phi$ of ϕ? It is immediate that if we extend periodically $scsin$ to the real line and do the same with $sqcos$, then

$$H(sqsin)(x) = \xi(x)\,sqcos(x)\,, \tag{9.1}$$

where ξ is a non-negative 1-periodic function equal to $\Xi(4^M x)$, and Ξ on $[0,1]$ looks as follows. It is logarithmically goes to $+\infty$ at 0, at $\frac{1}{2}-$, at $\frac{1}{2}+$ and at 1. It has two zeros: at $\frac{1}{4}$ and at $\frac{3}{4}$.

It is now tempting (looking at the definition of ϕ) to write that

$$H\phi(x) = \sum_{n=1}^{N}\sum_{I\in\mathcal{F},\ell(I)=4^{-n}}\sum_{J\,supervised\,by\,I} a_I\xi(x)sqc_J(x)\,.$$

Unfortunately, in (9.1) we have 1-periodic $sqsin, sqcos$ and not their localized to J's versions sqs_J, sqc_J. But H of any bounded highly oscillating function on interval J goes to zero uniformly outside the neighborhood of the end-points of J. There fore we can make up for the problem with localized to J functions sqs_J, sqc_J by

$$H\phi(x) = \sum_{n=1}^{N}\sum_{I\in\mathcal{F},\ell(I)=4^{-n}}\sum_{J\,supervised\,by\,I} a_I\xi(x)sqc_J(x) + \Theta(x)\,, \tag{9.2}$$

where $\Theta(x)$ is as close to zero as we wish on a set of as small Lebesgue measure as we wish–by the choice of M.

Function Ξ is a nice non-negative logarithmically blowing-up function on $I_0=[0,1]$. Let $\sigma := \int_0^1 \Xi(x)\,dx$. We can find the absolute constant a_0 such that

$$\Xi(x)\ge\sigma\,,\ \text{on a half of each quarter of } I_0\,. \tag{9.3}$$

So using 4) of Theorem 9.1 and recalling that the decomposition of ρ in it starts with constant g

$$-\sum_{n=1}^{N}\sum_{I\in\mathcal{F},\ell(I)=4^{-n}}\sum_{J\,supervised\,by\,I} a_I\xi(x)sqc_J(x) \ge \sigma g$$

on a set whose W-measure is greater than $\frac{1}{4}\,c_0\,Q\log^{1/7}Q\frac{1}{g}\int_{I_0}\Phi\,dx$.

Now compare this and (9.2) to see that for $\lambda := \frac{1}{2}\sigma g$

$$-H\phi(x)\ge\lambda \text{ on a set whose } W \text{ measure is greater than } \frac{1}{16}\,c_0\sigma\,Q\log^{1/7}Q\,\frac{1}{\lambda}\int_{I_0}\Phi\,dx\,. \tag{9.4}$$

Using 1) of Theorem 9.1 on all extremely small intervals (on which W is constant) we notice that $\int_{I_0}\Phi\,dx \ge \int_{I_0}|\phi|\,w\,dx$. We plug this into (9.4). We also notice that w is doubling with an absolute constant and dyadic A_1, so it is the usual A_1 with $[w]_{A_1}| \le B_0\, Q$ with absolute constant B_0. So the A_1 conjecture for the Hilbert transform with respect to the classical A_1 weights is disproved and we are done.

References

[1] A. Lerner, S. Ombrosi, C. Pérez, *A1 bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden*, Math. Res. Lett., **16** (2009) no. 1, 149-156.

[2] F. Nazarov, A. Volberg, *The Bellman function, the two-weight Hilbert transform, and embedding of the model spaces K_θ*, volume in honor of Thomas Wolff, J. d'Analyse Math. **87**, (2002), pp. 385–414.

[3] C. Prez, *Weighted norm inequalities for singular integral operators*. J. London Math. Soc. (2) **49** (1994), no. 2, 296308.

[4] M.C. Reguera, *On Muckenhoupt–Wheeden conjecture*, preprint, August, 2010, pp. 1–14.

[5] M.C. Reguera, C. Thiele, The Hilbert transform does not map $L^1(Mw)$ to $L^{1,\infty}(w)$. arXiv:1011.1767