BELLMAN FUNCTION, POLYNOMIAL ESTIMATES OF WEIGHTED DYADIC SHIFTS, AND A_2 CONJECTURE

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ABSTRACT. We give a short and simple polynomial estimate of the norm of weighted dyadic shift, which is linear in the norm of the weight.

1. INTRODUCTION

Recall that in [?] it was proved that

Theorem 1.1. *If T is an arbitrary operator with a Calderón–Zygmund kernel, then*

$$\|T\|_{L^2(w)\to L^{2,\infty}(w)}+\|T'\|_{L^2(w^{-1})\to L^{2,\infty}(w^{-1})}\le 2\|T\|_{L^2(w)\to L^2(w)}$$
$$\le C\left([w]_{A_2}+\|T\|_{L^2(w)\to L^{2,\infty}(w)}+\|T'\|_{L^2(w^{-1})\to L^{2,\infty}(w^{-1})}\right).$$

By T' we denote the adjoint operator. Here of course only the right inequality is interesting. And it is unexpected too. The weak and strong norm of any operator with a Calderón–Zygmund kernel turned out to be equivalent up to additive term $[w]_{A_2}$. From this we obtained in [?] the result which holds for any Calderón–Zygmund operator.

Theorem 1.2. $\|T\|_{L^2(w)\to L^2(w)}\le C\cdot[w]_{A_2}\log(1+[w]_{A_2})$.

By A_2 conjecture people understand the strengthening of this claim, where the logarithmic term is deleted, in other words, a linear (in weight's norm) estimate of arbitrary weighted Calderón–Zygmund operator. In [?] the A_2 conjecture was proved for Calderón–Zygmund operators having more than $2d$ smoothness in $\mathbb{R}^d$. Recently a preprint [?] of Tuomas Hytönen has appeared, the A_2 conjecture is fully proved there. It is based on the main theorem in the paper [?] of Pérez–Treil–Volberg. Both [?] and [?] are neither short nor easy.

The direct proof of A_2 conjecture (without going through [?]) was given in [?], and it was based on two ingredients: 1) a formula for decomposing an arbitrary Calderón–Zygmund operators into (generalized) dyadic shifts by the averaging trick, 2) on a polynomial in complexity and linear in weight estimate of the norm of a dyadic shift.

The latter was quite complicated and was based on modification of the argument in Lacey–Petermichl-Reguera [?]. The former was rooted in the works on non-homogeneous Harmonic Analysis, like e. g. [?]–[?], [?], but with a new twist, which appeared first in Hytönen's [?] and was simplified in Hytönen–Pérez–Treil–Volberg's [?].

The averaging trick was a development of the bootstrapping argument used by Nazarov–Treil–Volberg, where they exploited the fact that the bad part of a function can be made arbitrarily small. Using the original Nazarov–Treil–Volberg averaging trick would add an extra factor depending on $[w]_{A_2}$ to the estimate, so a new idea was necessary. The new observation in [?] was that as soon as the probability of a "bad" cube is less than 1, it is possible to completely ignore the bad cubes (at least in the situation where they cause troubles).

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2. SHIFTS OF COMPLEXITY m,n

We call by $\mathbb{S}_{m,n}$ the operator given by the kernel

$$f \to \sum_{L\in\mathcal{D}} \int_L a_L(x,y) f(y)dy\,,$$

where

$$a_L(x,y) = \sum_{\substack{I\subset L, J\subset L\\ \ell(I)=2^{-m}\ell(L),\, \ell(J)=2^{-n}\ell(L)}} c_{L,I,J} h_J(x) h_I(y)\,,$$

where h_I, h_J are Haar functions normalized in $L^2(dx)$ and $|c_{L,I,J}| \le \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$.

We are interested in sufficiently good estimate of

$$\|\mathbb{S}_{m,n}\|_w := \|\mathbb{S}_{m,n} : L^2(w) \to L^2(w)\|\,,$$

where $w \in A_2$. For such w we put $\sigma = w^{-1}$ and

$$[w]_{A_2} := \sup_I \langle w\rangle_I \langle\sigma\rangle_I < \infty\,,$$

and call it the norm of w (it is not a norm).

In recent paper [**?**] the following theorem was proved

Theorem 2.1.

$$\|\mathbb{S}_{m,n}\|_w \le C\,(m+n+1)^a [w]_{A_2}\,. \tag{2.1}$$

In [**?**] $a = 3$. Looks like here we have the same numerical value. But for its main application in [**?**]: the proof of A_2 conjecture, the value of a (if finite) is not important. The proof was hard and combinatorial, it was based on the ideas of [**?**], where such an estimate was proved with exponential dependence on $m+n$. We propose here a simple proof. This note is the fulfillment of the idea tried in preprint [**?**]. As the reader will see one needs a couple of new tricks to achieve this fulfillment.

Remark. The reasoning below is in $\mathbb{R}$. But one can modify it without any efforts to any $\mathbb{R}^d$. Moreover, in [**?**] the probability space of Christ's type dyadic lattices is built on any compact metric space with the property of geometric doubling (every ball contains at most a fixed number of disjoint balls of half a radius), which allows to extend the sharp bound of Calderón–Zygmund operators into metric space setting by repeating the averaging trick that reduces everything to the case of dyadic shift on the metric space, and then using this preprint to give a polynomial in complexity and linear in weight estimate for any shift.

3. THE HEART OF THE MATTER: A REDUCTION TO BILINEAR EMBEDDING ESTIMATE

To prove Theorem **??** we need the following decomposition:

Lemma 3.1.

$$h_I = \alpha_I h_I^w + \beta_I \frac{\chi_I}{\sqrt{I}}\,,$$

where

1) $|\alpha_I| \le \sqrt{\langle w\rangle_I}$*,*

2)$|\beta_I| \le \frac{|\Delta_I w|}{\langle w\rangle_I}$*,*

3) $\{h_I^w\}_I$ *is an orthonormal basis in* $L^2(w)$*,*

4) h_I^w *assumes on* I *two constant values, one on* I_+ *and another on* I_-*.*

Fix $\phi \in L^2(w), \psi \in L^2(\sigma)$. We need to prove

$$|(\mathbb{S}_{m,n}\phi w, \psi\sigma)| \le C\,(n+m+1)^a \|\phi\|_w \|\psi\|_\sigma\,. \tag{3.1}$$

We estimate $(\mathbb{S}_{m,n}\phi w, \psi\sigma)$ as

$$\begin{gathered}
|\sum_L\sum_{I,J} c_{L,I,J}(\phi w, h_I)(\psi\sigma, h_J)| \le \\
\sum_L\sum_{I,J} |c_{L,I,J}(\phi w, h_I^w)\sqrt{\langle w\rangle_I}(\psi\sigma, h_J^\sigma)|\sqrt{\langle\sigma\rangle_J}| + \\
\sum_L\sum_{I,J} |c_{L,I,J}\langle\phi w\rangle_I \frac{\Delta_I w}{\langle w\rangle_I}(\psi\sigma, h_J^\sigma)\sqrt{\langle\sigma\rangle_J}\sqrt{I}| + \\
\sum_L\sum_{I,J} |c_{L,I,J}\langle\psi\sigma\rangle_J \frac{\Delta_J\sigma}{\langle\sigma\rangle_J}(\phi w, h_I^w)\sqrt{\langle w\rangle_I}\sqrt{J}| + \\
\sum_L\sum_{I,J} |c_{L,I,J}\langle\phi w\rangle_I\langle\psi\sigma\rangle_J \frac{\Delta_I w}{\langle w\rangle_I}\frac{\Delta_J\sigma}{\langle\sigma\rangle_J}\sqrt{I}\sqrt{J}| =: I + II + III + IV\,.
\end{gathered}$$

We can notice that because $|c_{L,I,J}| \le \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ each sum inside L can be estimated by a perfect product of S and R terms, where

$$R_L(\phi w) := \sum_{I\subset L\dots} \langle\phi w\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I}\frac{|I|}{\sqrt{|L|}}$$

$$S_L(\phi w) := \sum_{I\subset L\dots} (\phi w, h_I^w)\sqrt{\langle w\rangle_I}\frac{\sqrt{|I|}}{\sqrt{|L|}}$$

and the corresponding terms for $\psi\sigma$. So we have

$$I \le \sum_L S_L(\phi w)S_L(\psi\sigma),\; II \le \sum_L S_L(\phi w)R_L(\psi\sigma),$$

$$III \le \sum_L R_L(\phi w)S_L(\psi\sigma),\;\; IV \le \sum_L R_L(\phi w)R_L(\psi\sigma)\,.$$

Now

$$S_L(\phi w) \le \sqrt{\sum_{I\subset L\dots} |(\phi w, h_I^w)|^2}\sqrt{\langle w\rangle_L},\; S_L(\psi\sigma) \le \sqrt{\sum_{J\subset L\dots} |(\psi\sigma, h_J^\sigma)|^2}\sqrt{\langle\sigma\rangle_L} \tag{3.2}$$

Therefore,

$$I \le C[w]_{A_2}^{1/2}\|\phi\|_w\|\psi\|_\sigma\,. \tag{3.3}$$

Terms II, III are symmetric, so consider III. Using Bellman function $(xy)^\alpha$ one can prove now

Lemma 3.2. *The sequence*

$$\mu_I := \langle w\rangle_I^\alpha\langle\sigma\rangle_I^\alpha\left(\frac{|\Delta_I w|^2}{\langle w\rangle_I^2} + \frac{|\Delta_I\sigma|^2}{\langle\sigma\rangle_I^2}\right)|I|$$

form a Carleson measure with Carleson constant at most $c_\alpha Q^\alpha$, where $Q := [w]_{A_2}$ for any $\alpha \in (0, 1/2)$.

For $S_L(\psi\sigma)$ we already had estimate (**??**).

To estimate $R_L(\phi w)$ let us denote by $\mathscr{P}_L$ maximal stopping intervals $K \in \mathscr{D}, K \subset L$, where the stopping criteria are 1) either $\frac{|\Delta_K w|}{\langle w\rangle_K} \ge \frac{1}{m+n+1}$, or $\frac{|\Delta_K\sigma|}{\langle\sigma\rangle_K} \ge \frac{1}{m+n+1}$, or 2) $\ell(K) = 2^{-m}\ell(L)$.

Lemma 3.3. *If K is any stopping interval then*

$$\sum_{I\subset K,\ell(I)=2^{-m}\ell(L)} |\langle\phi w\rangle_I|\frac{|\Delta_I w|}{\langle w\rangle_I}\frac{|I|}{\sqrt{|L|}} \le 2e^{\alpha}(m+n+1)\langle|\phi|w\rangle_K\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\,. \tag{3.4}$$

Proof. If we stop by the first criterion, then

$$\sum_{I\subset K,\ell(I)=2^{-m}\ell(L)} |\langle\phi w\rangle_I|\frac{|\Delta_I w|}{\langle w\rangle_I}\frac{|I|}{\sqrt{|L|}} \le 2\sum_{I\subset K,\ell(I)=2^{-m}\ell(L)}|\langle\phi w\rangle_I||I|\frac{1}{|K|}\frac{|K|}{\sqrt{|L|}} \le 2\,\langle|\phi|w\rangle_K\frac{|K|}{\sqrt{|L|}}$$

$$\le 2(m+n+1)\langle|\phi|w\rangle_K\left(\frac{|\Delta_K w|}{\langle w\rangle_K}+\frac{|\Delta_K\sigma|}{\langle\sigma\rangle_K}\right)\frac{|K|}{\sqrt{|L|}} \le 2(m+n+1)\langle|\phi|w\rangle_K\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}\langle w\rangle_K^{-\alpha/2}\langle\sigma\rangle_K^{-\alpha/2}\,.$$

Now replacing $\langle w\rangle_K^{-\alpha/2}\langle\sigma\rangle_K^{-\alpha/2}$ by $\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}$ does not grow the estimate by more than e^{α} as all pairs of son/father intervals larger than K and smaller than L will have there averages compared by constant at most $1\pm\frac{1}{m+n+1}$. And there are at most m such intervals between K and L.

If we stop by the second criterion, then K is one of I's, $\ell(I)=2^{-m}\ell(L)$ and

$$|\langle\phi w\rangle_I|\frac{|\Delta_I w|}{\langle w\rangle_I}\frac{|I|}{\sqrt{|L|}} \le |\langle\phi w\rangle_K|\frac{|K|}{\sqrt{|L|}}\frac{|\Delta_K w|}{\langle w\rangle_K} \le \langle|\phi|w\rangle_K\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}\langle w\rangle_K^{-\alpha/2}\langle\sigma\rangle_K^{-\alpha/2}\,.$$

Now we replace $\langle w\rangle_K^{-\alpha/2}\langle\sigma\rangle_K^{-\alpha/2}$ by $\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}$ as before.

□

Now

$$R_L(\phi w) \le C(m+n+1)\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\sum_{K\in\mathscr{P}_L}\langle|\phi|w\rangle_K\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}$$

$$\le C(m+n+1)\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\left(\sum_{K\in\mathscr{P}_L}\langle|\phi|w\rangle_K^2\frac{|K|}{|L|}\right)^{1/2}(\widetilde{\mu}(L))^{1/2}\,,$$

where

$$\widetilde{\mu}_L=\sum_{K\in\mathscr{P}_L}\mu_K\,.$$

Notice that $\widetilde{\mu}_L$ form a Carleson measure with constant at most $C(m+1)Q^{\alpha}$.

Now we make a trick! We will estimate the right hand side as

$$R_L(\phi w) \le C(m+n+1)\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\left(\sum_{K\in\mathscr{P}_L}\langle|\phi|w\rangle_K^p\frac{|K|}{|L|}\right)^{1/p}(\widetilde{\mu}_L)^{1/2}\,,$$

where $p=2-\frac{1}{m+n+1}$. In fact,

$$\left(\sum_{K\subset L,\,K\,is\,maximal}\langle|\phi|w\rangle_K^2\frac{|K|}{|L|}\right)^{p/2} \le \sum_{K\in\mathscr{P}_L}\langle|\phi|w\rangle_K^p\left(\frac{|K|}{|L|}\right)^{p/2}\,.$$

But $(2^{-j})^{1-\frac{1}{m+n+1}}\le 2\cdot 2^{-j}$ if $0\le j\le m$. So the trick is justified. Therefore, using Cauchy inequality, one gets

$$R_L(\phi w) \le C(m+n+1)\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\left(\sum_{K\in\mathscr{P}_L}\langle|\phi|^p w\rangle_K\langle w\rangle_K^{p-1}\frac{|K|}{|L|}\right)^{1/p}(\widetilde{\mu}_L)^{1/2}\,.$$

We can replace all $\langle w\rangle_K^{p-1}$ by $\langle w\rangle_L^{p-1}$ paying the price by constant. This is again because all intervals larger than K and smaller than L will have there averages compared by constant at most $1\pm\frac{1}{m+n+1}$. And there are at most m such intervals between K and L. Finally,

$$R_L(\phi w)\le C(m+n+1)\langle w\rangle_L^{-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\left(\sum_{K\in\mathscr{P}_L}\langle|\phi|^p w\rangle_K\frac{|K|}{|L|}\right)^{1/p}\langle w\rangle_L^{1-\frac{1}{p}}(\widetilde{\mu}_L)^{1/2} \tag{3.5}$$

We need the standard notations: if μ is an arbitrary positive measure we denote

$$M_\mu f(x):=\sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(x)|\,d\mu(x)\,.$$

In particular M_w will stand for this maximal function with $d\mu=w(x)\,dx$.

From (**??**) we get

$$R_L(\phi w)\le C(m+n+1)\langle w\rangle_L^{1-\alpha/2}\langle\sigma\rangle_L^{-\alpha/2}\inf_L M_w(|\phi|^p)^{1/p}(\widetilde{\mu}_L)^{1/2} \tag{3.6}$$

Now

(3.7)

$$S_L(\psi\sigma)R_L(\phi w)\le C(m+n+1)\langle w\rangle_L^{1-\alpha/2}\langle\sigma\rangle_L^{1-\alpha/2}\frac{\inf_L M_w(|\phi|^p)^{1/p}}{\langle\sigma\rangle_L^{1/2}}(\widetilde{\mu}_L)^{1/2}\sqrt{\sum_{J\subset L\dots}|(\psi\sigma,h_J^\sigma)|^2}\,,$$

$$R_L(\psi\sigma)R_L(\phi w)\le C(m+n+1)\langle w\rangle_L^{1-\alpha}\langle\sigma\rangle_L^{1-\alpha}\inf_L M_w(|\phi|^p)^{1/p}\inf_L M_\sigma(|\psi|^p)^{1/p}\widetilde{\mu}_L\,. \tag{3.8}$$

Now we use the Carleson property of $\{\widetilde{\mu}_L\}_{L\in\mathscr{D}}$. We need a simple folklore Lemma.

Lemma 3.4. *Let $\{\alpha_L\}_{L\in\mathscr{D}}$ define Carleson measure with intensity B. Let F be a positive function on the line. Then*

$$\sum_L(\inf_L F)\,\alpha_L\le 2B\int_{\mathbb{R}}F\,dx\,. \tag{3.9}$$

$$\sum_L\frac{\inf_L F}{\langle\sigma\rangle_L}\alpha_L\le CB\int_{\mathbb{R}}\frac{F}{\sigma}dx\,. \tag{3.10}$$

Now use (**??**). Then the estimate of $III\le\sum_L S_L(\psi\sigma)R_L(\phi w)$ will be reduced to estimating

$$(m+n+1)Q^{1-\alpha/2}\left(\sum_L\frac{\inf_L M_w(|\phi|^p)^{2/p}}{\langle\sigma\rangle_L}\widetilde{\mu}_L\right)^{1/2}\le(m+n+1)^2Q\left(\int_{\mathbb{R}}(M_w(|\phi|^p))^{2/p}wdx\right)^{1/2}$$

$$\le(\frac{1}{2-p})^{1/p}(m+n+1)^2Q\left(\int_{\mathbb{R}}\phi^2\,wdx\right)^{1/2}\le(m+n+1)^3Q\left(\int_{\mathbb{R}}\phi^2\,wdx\right)^{1/2}.$$

Here we used (**??**) and the usual estimates of maximal function M_μ in $L^q(\mu)$ when $q\approx 1$. Of course for II we use the symmetric reasoning.

Now IV: we use (**??**) first.

$$\sum_L S_L(\psi\sigma)R_L(\phi w)\le(m+n+1)Q^{1-\alpha}\sum_L\inf_L M_w(|\phi|^p)^{1/p}\inf_L M_\sigma(|\psi|^p)^{1/p}\widetilde{\mu}_L$$

$$\le C(m+n+1)^2Q\int_{\mathbb{R}}(M_w(|\phi|^p))^{1/p}\,(M_\sigma(|\psi|^p))^{1/p}w^{1/2}\sigma^{1/2}dx$$

$$\le C(m+n+1)^2Q\left(\int_{\mathbb{R}}(M_w(|\phi|^p))^{2/p}\,wdx\right)^{1/2}\left(\int_{\mathbb{R}}(M_\sigma(|\psi|^p))^{2/p}\,\sigma dx\right)^{1/2}$$

$$\leq C(m+n+1)^4 Q\left(\int_{\mathbb{R}} \phi^2 w dx\right)^{1/2}\left(\int_{\mathbb{R}} \psi^2 \sigma dx\right)^{1/2}.$$

Here we used (**??**) and the usual estimates of maximal function M_μ in $L^{2/p}(\mu)$ when $p \approx 2$, $p < 2$.

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