ON WEAK AND STRONG SHARP WEIGHTED ESTIMATES OF
SQUARE FUNCTION
PAATA IVANISVILI, FEDOR NAZAROV, AND ALEXANDER VOLBERG

Abstract. We reduce here end-point estimates for one singular operator (namely for dyadic square function) to Monge–Ampère equations with drift. The spaces are weighted spaces, and therefore the domain, where we solve our PDE is non-convex. If we are in the end-point situation our goal is either to find a logarithmic blow-up of the norm estimate from below, or to prove that there is upper estimate of the norm without logarithmic blow-up. For two other singular operators: the martingale transform and the Hilbert transform the logarithmic blow-up was demonstrated in [5]. This disproved the so-called weak Muckenhoupt conjecture, while [6] and [7] disproved the strong Muckenhoupt conjecture for these operators. The difference with the current situation is that for the martingale transform and for the Hilbert transform (for their weak type estimates in weighted spaces) the end-point exponent is 1, interesting weights were $A_1$ weights, and the abovementioned logarithmic blow-up concerns $\log[w]_{A_1}$. For the weak type estimate of the square function the end-point exponent is 2, and the abovementioned logarithmic blow-up concerns $\log[w]_{A_2}$. We show that for weak type test condition there is no logarithmic blow-up, which is strikingly different from the upper estimate in [3] and [1], where logarithm of $[w]_{A_2}$ is present.

1. Notations and results

We reduce here end-point estimates for one singular operator (namely for dyadic square function) to Monge–Ampère equations with drift. The spaces are weighted spaces, and therefore the domain, where we solve our PDE is non-convex. If we are in the end-point situation our goal is either to find a logarithmic blow-up of the norm estimate from below, or to prove that there is upper estimate of the norm without logarithmic blow-up. For two other singular operators: the martingale transform and the Hilbert transform the logarithmic blow-up was demonstrated in [5]. This disproved the so-called weak Muckenhoupt conjecture, while [6] and [7] disproved the strong Muckenhoupt conjecture for these operators. (Disproving a weak conjecture is a stronger result than disproving the strong conjecture.)

The difference with the current situation is that for the martingale transform and for the Hilbert transform (for their weak type estimates in weighted spaces) the end-point exponent is 1, interesting weights were $A_1$ weights, and the mentioned logarithmic blow-up concerns $\log[w]_{A_1}$. For the weak type estimate of the square function the end-point exponent is 2, and the above mentioned logarithmic blow-up concerns $\log[w]_{A_2}$. We show that for weak type test condition there is no logarithmic blow-up, which is strikingly different from the upper estimate in [3] and [1], where logarithm of $[w]_{A_2}$ is present. In the paper [3] the weak weighted norm of the square function from $L^2(w)$ to $L^{2,\infty}(w)$ was estimated from above by $\sqrt{[w]_{A_2}} \log[w]_{A_2}$, this was improved to $\sqrt{[w]_{A_2}} \sqrt{\log[w]_{A_2}}$.

2010 Mathematics Subject Classification. 42B20, 42B35, 47A30.
FN is partially supported by the NSF grant DMS.
AV is partially supported by the NSF grant DMS-1265549 and by the Hausdorff Institute for Mathematics, Bonn, Germany.
It is easy to conjecture that this estimate is sharp. However, below we show that if one tests the norm on characteristic functions of cubes (intervals) then logarithmic term disappears. In particular, if there were a T1 theorem for weak type theorems, then we would conclude that logarithmic term totally disappears, and one would be able to improve the estimates \( \sqrt{[w]_{A_2}} \sqrt{\log [w]_{A_2}} \) of [3], [1] to a simple \( \sqrt{[w]_{A_2}} \) (which cannot be improved by an obvious examples of one-singular-point weights).

It is interesting that for the square function estimate the end-point exponent is 2 and one should work with \( A_2 \) weights. The square function operator is much less singular than the martingale transform and the Hilbert transform. This is why the game happens around \( \sqrt{[w]_{A_2}} \). We recall the reader that for the martingale transform and the Hilbert transform the play happens around \( [w]_{A_1} \) and the end-point exponent is 1. Let us recall that for those operators the estimate from below logarithmically blows up. It has been shown in [5] that \( \| L^1(w) \to L^{1, \infty}(w) \| \)-norm of both martingale and Hilbert transform can have the estimate from below of the order \( [w]_{A_1} (\log [w]_{A_1})^\varepsilon \), for \( \varepsilon > 0 \). In fact, it has been shown that any \( \varepsilon \in (0, 0.2) \) would work (and \( \varepsilon \in (0, 2/7) \) would work for another type of martingale transform).

In what follows, given \( w \in A_2 \), we consider for constants: 1) \( \| L^2(w) \to L^{2, \infty}(w) \| \)-norm of the square function operator on test functions, 2) the full \( \| L^2(w) \to L^{2, \infty}(w) \| \)-norm of the square function operator, 3) \( \| L^2(w) \to L^2(w) \| \)-norm of the square function operator on test functions, 4) the full \( \| L^2(w) \to L^2(w) \| \)-norm of the square function operator. The strong norms in 3) and 4) are known of course, and we include them into consideration by two reasons. The first reason is that by Chebyshev inequality one can estimate the weak norms by the strong ones, and the second reason is to show how useless they are for the sharp weak estimate via Chebyshev inequality, they are of order \( [w]_{A_2} \), while the weak estimates are small perturbations of \( \sqrt{[w]_{A_2}} \).

One disclaimer: the reader should notice that everywhere below we are working not with norm (strong or weak) but with squares of the norms.

1.1. **Four constants.** We consider the square function transform. To do that we consider the dyadic (for simplicity) lattice \( \mathcal{D} \) and call its elements by \( I, J, \ldots \). We consider martingale differences (the symbol \( \text{ch}(J) \) denotes the dyadic children of \( J \))

\[
\Delta_J f := \sum_{I \in \text{ch}(J)} \chi_I (\langle f \rangle_I - \langle f \rangle_J).
\]

In what follows the symbol \( \langle f \rangle_I \) denotes average value of \( f \) over the set \( I \) i.e., \( \langle f \rangle = \frac{1}{|I|} \int_I f \).

For the simplest case of dyadic lattice on the line we have that \( |\Delta_J f| \) is constant on \( J \), and

\[
\Delta_J f = \frac{1}{2} [\langle w \rangle_{J_+} - \langle w \rangle_{J_-}] \chi_{J_+} + [\langle w \rangle_{J_-} - \langle w \rangle_{J_+}] \chi_{J_-}.
\]

The square function transform operator is

\[
(Sf)^2(x) = \sum_{J \in \mathcal{D}} |\Delta_J f|^2 \chi_J(x).
\]

Everywhere above \( f \) is a function with values in \( \mathbb{C} \). We consider a positive function \( w(x) \), and we call it \( A_2 \) weight if
ON WEAK AND STRONG END-POINT SHARP WEIGHTED ESTIMATES OF SQUARE FUNCTION

(1.1) \[ Q := \sup_{J \in \mathcal{D}} \langle w \rangle_J \langle w^{-1} \rangle_J < \infty. \]

In this note we wish to compare the sharp weighted estimates for square function, weak and strong. In fact, we are interested in two weak estimates like those:

(1.2) \[ \frac{1}{|J|} w \left\{ x \in J : \sum_{I \in D(J)} |\Delta_I w^{-1}|^2 \chi_I(x) > \lambda \right\} \leq C_{w,T}(Q) \frac{\langle w^{-1} \rangle_J}{\lambda}. \]

(1.3) \[ \frac{1}{|J|} w \left\{ x \in J : \sum_{I \in D(J)} |\Delta_I (f w^{-1})|^2 \chi_I(x) > \lambda \right\} \leq C_w(Q) \frac{\langle f^2 w^{-1} \rangle_J}{\lambda}. \]

We wish to understand the sharp order of magnitude of constants \( C_{w,T}, C_w \) in terms of \( Q \) (\( Q \) is large). We wish to compare them to the similar estimates of strong type

(1.4) \[ \frac{1}{|J|} \sum_{I \in D(J)} |\Delta_I w^{-1}|^2 \langle w \rangle_I |I| \leq C_{s,T}(Q) \langle w^{-1} \rangle_J. \]

(1.5) \[ \frac{1}{|J|} \sum_{I \in D(J)} |\Delta_I (f w^{-1})|^2 \langle w \rangle_I |I| \leq C_s(Q) \langle f^2 w^{-1} \rangle_J. \]

Of course, there are several obvious estimates: 1) weak constants are smaller than strong constants:

\[ C_{w,T} \leq C_{s,T}, \quad C_w \leq C_s \]
just by Chebyshev inequality; also 2) test function estimates are trivially at least as good than the general estimates:

\[ C_{w,T} \leq C_w, \quad C_{s,T} \leq C_s. \]

It is quite well known (although far from being trivial) that there is at least one converse inequality:

(1.6) \[ C_s \leq A(d) C_{s,T}. \]

This is an instance of the celebrated \( T1 \) theorem, on this occasion applied to square function transform in the weighted situation.

Another converse estimate

(1.7) \[ C_w \leq A(d) C_{w,T} \]
is a puzzle for us, we could not find in the literature such an estimate (we can call it weak norm \( T1 \) theorem), and we do not know whether and when it can be true.

The goal of this paper is a) to give a sharp estimate of \( C_{w,T} \), b) to give a sharp estimate of \( C_{s,T} \) (and thus of \( C_s \), see (1.6)), c) to discuss the sharpness of the estimate \( C_w \) (first done by [3] and improved by [1]).

Remark 1. We are working with operator \( S : L^2(w) \to L^{2,\infty}(w) \) or \( S : L^2(w) \to L^2(w) \). However, it is more convenient to work with isomorphic objects: \( S_{w^{-1}} : L^2(w^{-1}) \to L^{2,\infty}(w) \) or \( S_{w^{-1}} : L^2(w^{-1}) \to L^2(w) \), here \( S_{w^{-1}} \) denotes the product \( SM_{w^{-1}} \), where \( M_{w^{-1}} \) is the operator of multiplication.
Remark 2. The reader can see that (1.2) estimate is a particular case of (1.3) estimate for a special choice of test function \( f = \chi_J \). The same remark holds for (1.4) and (1.5). We already noted above that test function estimate (1.4) (plus its symmetric counterpart for \( w \) exchanged by \( w^{-1} \)) imply the strong estimate (1.5). This, as we already mentioned, is the essence of weighted \( T_1 \) theorem (testing condition theorem in the terminology of E. Sawyer, but for singular integrals).

However, to the best of our knowledge, for weak estimate no \( T_1 \) type result like (1.7) (for weighted situation) is known.

2. Sharp constant in weak testing estimate

Theorem 2.1. \( C_{w,T} \leq AQ \) and this estimate is sharp.

We will prove the estimate, the sharpness is well known just for one-point-singularity weights.

We introduce the following function of 3 real variables

\[
B(u, v, \lambda) := \sup \frac{1}{|J|} w \left\{ x \in J : \sum_{I \in D(J)} |\Delta_I w^{-1}|^2 \chi_I(x) > \lambda \right\},
\]

where supremum is taken over all \( w \in A_2, [w]_{A_2} \leq Q \), such that

\[
\langle w \rangle_J = u, \langle w^{-1} \rangle_J = v.
\]

Notice that by scaling argument our function does not depend on \( J \) but depends on \( Q = [w]_{A_2} \).

Remark. Ideally we want to find the formula for this function. Notice that this is similar to solving a problem of "isoperimetric" type, where the solution of certain non-linear PDE is a common tool.

2.1. Properties of \( B \) and the main inequality. Notice several properties of \( B \):

- \( B \) is defined in \( \Omega := \{(u, v, \lambda) : 1 \leq uv \leq Q, u > 0, v > 0, 0 \leq \lambda < \infty \} \).
- If \( P = (u, v, \lambda), P^+ = (u+, v+, \lambda+), P^- = (u-, v-, \lambda-) \) belong to \( \Omega \), and \( u = \frac{1}{2}(u + u-), v = \frac{1}{2}(v + v-), \lambda = \min(\lambda+, \lambda-) \), then the main inequality holds with constant \( c = 1 \):

\[
B \left( u, v, \lambda + c(v + v-) \right) - \frac{1}{2} (B(P^+) + B(P^-)) \geq 0.
\]

- \( B \) is decreasing in \( \lambda \).
- Homogeneity \( B(ut, v/t, \lambda t^2) = tB(u, v, \lambda), t > 0 \).
- Obstacle condition: for all points \( (u, v, \lambda) \) such that \( 10 \leq uv \leq Q, \lambda \geq 0 \), if \( \lambda \leq a v^2 \) for some \( a \), then \( B(u, v, \lambda) = u \).
- The boundary condition \( B(u, v, \lambda) = 0 \) if \( uv = 1 \).

All these properties are very simple consequences of the definition of \( B \). However, let us explain a bit the second and the fifth bullet. The second bullet is the consequence of the scale invariance of \( B \). We consider data \( P^+ \) and find weight \( w^+ \) that almost supremizes \( B(P^+) \), we suppose to have it on \( J^+ \). But by scale invariance we can think that \( w^+ \) lives on \( J^+ \). Then we consider data \( P^- \) and find weight \( w^- \) that almost supremizes \( B(P^-) \), we suppose
to have it on $J$. But by scale invariance we can think that $w$ lives on $J$. The next step is to consider the concatenation of $w^+$ and $w^-$:

$$w_c := \begin{cases} w^+, & \text{on } J^+ \\ w^-, & \text{on } J^- \end{cases}$$

Clearly this new weight is a competitor for giving the supremum for date $P$ on $J$. But it is only a competitor, the real supremum in (2.1) is bigger. This implies the second bullet above (the main inequality).

Now let us explain the fifth bullet above, we call it the obstacle condition. Let us consider a special weight $w_s$ in $J$: it is one constant on $J^-$ and just another constant on $J^+$. Moreover, we wish to have $\langle w^{-1}_s \rangle_{J^+} = 4 \langle w^{-1}_s \rangle_{J^-}$. Notice that then $b\langle w^{-1}_s \rangle_J \leq |\Delta_J w^{-1}_s|$ with some positive absolute constant $b$. Now it is obvious that if $\lambda \leq b^2 \langle w^{-1}_s \rangle_J^2$ then $\{x \in J : S^2 w^{-1}_s(x) \geq \lambda\} = J$ and so $\frac{1}{|J|} w_s \{x \in J : S^2 w^{-1}_s(x) \geq \lambda\} = \langle w_s \rangle_J$. Notice now that $w_s$ is just one admissible weight, and that we have to take supremum over all such admissible weights. This gives the fifth bullet above (= the obstacle condition): $B(u, v, \lambda) = u$ for those points $(u, v, \lambda)$ in the domain of definition of $B$, where the corresponding $w_s$ with $\langle w_s \rangle = u, \langle w^{-1}_s \rangle_J = v$ exists. It is obvious that for all sufficiently large $Q = \langle w \rangle_{A_2}$ and for any pair $(u, v)$ such that $10 \leq uv \leq Q$ one can construct $w_s$ as above.

Notice that the main inequality above transforms into a partial differential inequality if considered infinitesimally (and if we tacitly assume that $B$ is smooth):

(2.2) $$\frac{1}{2} d^2 B - c \frac{\partial B}{\partial \lambda} (dv)^2 \leq 0.$$ We get it with $c = 1$ for the function $B$ defined above (if $B$ happens to be smooth).

We are not going to find $B$ defined in (2.1), but instead we will give $B$ that satisfies all the properties above (and of course (2.2)) except for the boundary condition (the last bullet above). It will satisfy even a slightly stronger properties, for example, the obstacle condition (the fifth bullet) will be satisfied with 1 instead of 10:

(2.3) $\forall (u, v, \lambda)$ such that $1 \leq uv \leq Q$, $\lambda \geq 0$, if $\lambda \leq a v^2$ for some $a$, then $B(u, v, \lambda) = u$.

Here $a$ will be some positive absolute constant (it will not depend on $Q$).

It is very easy to prove the following

**Theorem 2.2.** Suppose we have a function $B$ satisfying all the conditions above except the boundary condition, but satisfying the obstacle condition in the form (2.3). We also allow $c$ to be a small positive constant (say, $c = \frac{1}{2}$). And suppose it also satisfies

(2.4) $$B(u, v, \lambda) \leq A \frac{Q v}{\lambda}.$$ Then the constant $C_{w, T}$ in (1.2) is at most $A Q$.

**Proof.** It is an easy stopping time reasoning. It is enough to think that $w$ is constant on some very small dyadic intervals and to prove the estimate on $C_{w, T}$ uniformly. Then we start with any such $w$, $\langle w \rangle_{A_2} \leq Q$, and we use the main inequality with $u = \langle w \rangle_J, v = \langle w^{-1} \rangle_J, u\pm = \langle w \rangle_{J \pm}, v\pm = \langle w^{-1} \rangle_{J \pm}$, $\lambda = \langle w^{-1} \rangle_{J^+} - \langle w^{-1} \rangle_{J^-}$:

$$\frac{A Q \langle w^{-1} \rangle_J}{\lambda} \geq B(\langle w \rangle_J, \langle w^{-1} \rangle_J, \lambda) \geq \frac{1}{2} B(\langle w \rangle_{J^+}, \langle w^{-1} \rangle_{J^-}, \lambda^+) + B(-) \geq \cdots$$
We continue to use the main inequality (because \( J^\pm \) are not different from \( J \)) and finally after large but finite number of steps, on certain small intervals \( I = J^{\pm \cdots \pm} \) we come to the situation that

\[
\lambda^{\pm \cdots \pm} < c\langle (w^{-1})_{J^{\pm \cdots \pm}} \rangle^2 \leq a\langle (w^{-1})_{J^{\pm \cdots \pm}} \rangle^2.
\]

This happens exactly on those final intervals \( I \) on which the following holds for \( x \in I \):

\[
c \sum_{L \in D(J), I \subset L} |\Delta I w^{-1}|^2 \chi_L(x) > \lambda.
\]

At this moment we use the property 5 of \( B \) called obstacle condition. On such intervals (call them final intervals) \( I \) the obstacle condition will provide us with \( B(...I...) = \langle w \rangle_I \). So we get, on a certain large step \( N \) the following estimate

\[
\frac{1}{2^N} \sum_{I \in D_N(J), I \text{ is final}} \langle w \rangle_I \leq \frac{AQ \langle w^{-1} \rangle_J}{\lambda}.
\]

But for each final interval \( I \) we have \( 2^{-N} = I/J \). Therefore, we proved

\[
\frac{1}{J} \{ x \in J : \sum_{I \in D(J)} |\Delta I w^{-1}|^2 \chi_I(x) > \lambda \} \leq AQ \langle w^{-1} \rangle_J \lambda,
\]

which is (1.2).

\[\square\]

2.2. Formula for the function \( B \). Monge–Ampère equation with a drift. Here is the formula for \( B \) that satisfies all the properties in 2.1 (except for the last one, the boundary condition):

\[
B(u, v, \lambda) = \frac{1}{\sqrt{\lambda}} \Theta(\sqrt{\lambda} u, \frac{v}{\sqrt{\lambda}}), \text{ where } \Theta(\gamma, \tau) := \min \left( \gamma, Q e^{-\tau^2/2} \int_0^\tau e^{s^2/2} ds \right).
\]

Notice that the fact that \( B \) has the form \( B(u, v, \lambda) = \frac{1}{\sqrt{\lambda}} \Theta(\sqrt{\lambda} u, \frac{v}{\sqrt{\lambda}}) \) is trivial, this follows from property 4 called homogeneity. Notice also that function \( \Theta \) is also given in “hyperbolic domain” \( G := \{ (\gamma, \tau) > 0 : 1 \leq \gamma \tau \leq Q \} \).

All properties (except for the boundary condition, but we do not use it) follow by direct computation. In the next section we explain how to get this formula.

2.3. Explanation of how to find such a function \( \Theta \). The main inequality (with \( c = \frac{1}{8} \)) in terms of \( \Theta \) becomes a “drift concavity condition”:

\[
\frac{1}{\sqrt{1 + (\Delta \tau)^2/8}} \Theta\left( \sqrt{1 + (\Delta \tau)^2/8} \frac{\gamma_1 + \gamma_2}{2}, \frac{1}{\sqrt{1 + (\Delta \tau)^2/8}} \frac{\gamma_1 + \gamma_2}{2} \right) \geq \frac{1}{2} \left( \Theta(\gamma_1, \tau_1) + \Theta(\gamma_2, \tau_2) \right),
\]

where \((\gamma_1, \tau_1), (\gamma_2, \tau_2) \in G, 0 < \tau_1 < \tau_2, \Delta \tau := \tau_2 - \tau_1 \).

Assuming that \( \Theta \) is smooth (we will find a smooth function), the infinitesimal version appears, it is a sort of Monge–Ampère relationship with a drift. Namely, the following matrix relationship must hold

\[
\begin{bmatrix} \Theta_{\gamma \gamma} & \Theta_{\gamma \tau} \\ \Theta_{\tau \gamma} & \Theta_{\tau \tau} + \Theta + \tau \Theta_{\tau} - \gamma \Theta_{\gamma} \end{bmatrix} \leq 0.
\]
A direct calculation shows that this property is equivalent to the following one: On any piece of the curve $\gamma = \phi(\tau)$ lying in the domain $G$ such that

$$\phi'' + \tau \phi' + \phi = 0$$

we have $(\Theta(\phi(\tau), \tau)')'' + \tau(\Theta(\phi(\tau), \tau))' + \Theta(\phi(\tau), \tau) \leq 0$.

This hints at a possibility to have such a change of variables $(\gamma, \tau) \to (\Gamma, T)$ such that condition (2.7) transforms to a simple concavity. To some extent this is what happens. Namely, notice the following simple Lemma 2.3.

Consider the following change of variable:

$$T = \int_0^\tau e^{s^2/2} ds.$$ Then

$$\phi''(\tau) + \tau \phi'(\tau) + \phi(\tau) \leq 0$$

if and only if $(e^{\tau^2/2} \Theta(\phi(\tau), \tau))_{TT} \leq 0$.

The proof is a direct differentiation.

This Lemma immediately gives the following interpretation of (2.8): let us make the following change of variables in $\mathbb{R}_+^1 \times \mathbb{R}_+^1$:

$$\begin{cases} 
\Gamma := \gamma e^{\tau^2/2}, \\
T = \int_0^\tau e^{s^2/2} ds,
\end{cases}$$

then in the new coordinates any curve $\gamma = \phi(\tau)$ such that $\phi'' + \tau \phi' + \phi = 0$ becomes a straight line $\Gamma = CT + D$, and the condition $(\Theta(\phi(\tau), \tau))'' + \tau(\Theta(\phi(\tau), \tau))' + \Theta(\phi(\tau), \tau) \leq 0$ on this curve (line) becomes

$$(e^{\tau^2/2} \Theta(\phi(\tau), \tau))_{TT} \leq 0.$$ In other words, this means that the function

$$\Phi(\Gamma, T) := \Theta(\gamma, \tau)$$

satisfies the concavity condition

$$(U(T) \Phi(\Gamma, T))_{TT} \leq 0$$

on each line $\Gamma = CT + D$, where $U(T) := T'(\tau(T)) = e^{\tau^2/2}$.

This is just a concavity of $U(T) \Phi(\Gamma, T)$ of course.

So we reduce the question to finding a concave function in new coordinates. Now we choose a simplest possible concave function:

$$U(T) \Phi(\Gamma, T) := \min(\Gamma, QT).$$

If we write down now $\Theta(\gamma, \tau) = \Phi(\Gamma, T)$ in the old coordinates, we get exactly function $\Theta$ from (2.5), namely,

$$\Theta(\gamma, \tau) := \min \left( \gamma, Qe^{-\tau^2/2} \int_0^\tau e^{s^2/2} ds \right).$$

Recall that now we can consider

$$B(u, v, \lambda) = \frac{1}{\sqrt{\lambda}} \Theta(u\sqrt{\lambda}, \frac{v}{\sqrt{\lambda}})$$

and we are going to apply Theorem 2.2 to it.

First of all it is now very easy to understand why the form of the domain $G = \{1 \leq \gamma \tau \leq Q\}$ plays the role. In this domain our function $\Theta$ satisfies the obstacle condition

$$\Theta(\gamma, \tau) = \gamma$$

as soon as $\tau \geq a_0 > 0$,

where $a_0$ is an absolute positive constant.
Secondly, function $\Theta$ satisfies the infinitesimal condition (2.7) by construction. But we need to check that the main inequality (2.6) is satisfied as well.

This can be done by the following simple lemma.

**Lemma 2.4.** Inequality (2.6) for function $\Theta$ built above holds if and only if the following inequality is satisfied for $\phi(\tau) := e^{-\tau^2/2} \int_0^\tau e^{s^2/2} ds$:

\[
\frac{1}{\sqrt{1 + (\Delta \tau)^2/8}} \frac{\tau_1 + \tau_2}{2 \sqrt{1 + (\Delta \tau)^2/8}} \geq \frac{1}{2} (\phi(\tau_1) + \phi(\tau_2)), \forall 0 < \tau_1 \leq \tau_2 \leq \tau_0,
\]

with some absolute small $\tau_0$.

Lemma is easy, because we can immediately see that the main inequality (2.6) commutes with the operation of minimum. Inequality (2.14) is a direct if tedious calculation.

3. **Strong estimate for test functions**

**Theorem 3.1.** $C_{s,T} \leq AQ^2$ and this estimate is sharp.

We will adopt the same approach as in the previous section, but it will bring us to non-linear ODE rather than non-linear PDE.

First of all we wish to find a smooth $B(u, v)$ in the domain

$$O := O_Q := \{(u, v) > 0 : 1 \leq uv \leq Q\},$$

such that

\[
\frac{1}{2} d^2 B + u(dv)^2 \leq 0.
\]

We are searching for homogeneous $B$: $B(u/t, tv) = tB(u, v)$. Hence

$$B(u, v) = \frac{1}{u} \phi(uv)$$

Then (3.1) becomes

\[
\left[ x^2 \phi''(x) - 2x \phi'(x) + 2\phi, \frac{x \phi''(x)}{\phi''(x) + 2} \right] \leq 0.
\]

To have this it is enough to satisfy for all $x \in [1, Q]$

\[
\phi''(x) + 2 \leq 0, -x \phi'(x) + \phi(x) \leq 0, \phi'' \cdot (-x \phi' + \phi) + x^2 \phi'' - 2x \phi' + 2\phi = 0
\]

The last equation is just making the determinant of our matrix to vanish. Let us start with this equation and put $g = \phi(x)/x$. Then we know that $-x^2 g' = \phi - x \phi' \leq 0$, so $g$ is increasing. Also $xg'' + 2g' = \phi'' \leq -2$, hence $g'' \leq 0$ (we are on $[1, Q]$).

In terms of $g$ we have equation

$$x(-g'' + g') - 2(g')^2 = 0.$$

This is a first order non-linear ODE on $h := g'$ of which we know that $h \geq 0, h' \leq 0$:

$$x(-hh' + h') - 2h^2 = 0.$$

Variables separate and we get

\[
\frac{1 - \frac{h}{h^2} h'}{h^2} = \frac{2}{x}.
\]

The condition $\phi'' + 2 \leq 0$ is the same as $h \geq 1$, and the condition $\phi - x \phi' \leq 0$ is the same as $h \geq 0$. Thus any solution $h \geq 1$ of (3.3) gives the desired result.
We want to solve this for \( x \in [1, Q] \):
\[
- \log h - \frac{1}{h} = 2 \log x + c \quad \Leftrightarrow \quad - \frac{1}{h} e^{-\frac{1}{h}} = -x^2 C, \quad C > 0.
\]

Notice that Lambert \( W \) function (which is multivalued) solves the equation \( z = W(z) e^{W(z)} \). Thus we must have \( W(-x^2 C) = -\frac{1}{h(x)} \). The condition \( 0 \geq -\frac{1}{h(x)} \geq -1 \) requires that \( 0 \geq W \geq -1 \) and this gives single-valued function \( W_0(y) \) defined on the interval \([-1/e, 0]\) such that \( W_0(-1/e) = -1, W_0(0) = 0 \) and \( W_0(y) \) is increasing. So \( h(x) = -\frac{1}{W_0(-x^2 C)} \). The condition \(-1/e \leq -x^2 C \leq 0\) for \( x \in [1, Q] \) gives the range for constant \( C \) i.e., \( 0 < C \leq \frac{1}{Q^2 e} \).

Going back to the functions \( \phi \) and \( B \) we obtain:
\[
\varphi(x) = -x \int_1^x \frac{dt}{W_0(-t^2 C)} + x \varphi(1) \quad \text{and} \quad B(u, v) = -v \int_1^{uv} \frac{dt}{W_0(-t^2 C)} + v \varphi(1).
\]

It is not difficult to see that the function \( B := B_{4Q} \) satisfies a better estimate called the main inequality for strong test estimate:

\[ B(u, v) - \frac{1}{2} (B(u+, v+) + B(u-, v-)) \geq c(v + v-)^2 \]

with small positive absolute \( c \), whenever \( (u, v), (u+, v+), (u-, v-) \in O_Q \) and \( (u, v) = \frac{1}{2}((u+, v+) + (u-, v-)) \).

Let us also see how bounded is \( B \). Choosing \( C = \frac{1}{Q^2 e} \) gives minimal \( B \). \( B(u, v) \) can be assumed to be 0 if \( uv = 1 \). Hence \( \phi(1) = 0 \). Then

\[ B(u, v) = -v \int_1^{uv} \frac{dt}{W_0(-t^2 C)} \leq v \int_1^{uv} \frac{Q^2 e}{t^2} = Q^2 e v \left(1 - \frac{1}{uv}\right), 1 \leq uv \leq Q. \]

Here we used the fact that \( W_0(x) \leq x \) for \( x \in [-1/e, 0] \). Actually one can get better estimates by using the series expansion for \( W_0 \) i.e.,
\[
W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n, \quad |x| < \frac{1}{e}.
\]

Emulating almost verbatim the proof of Theorem 2.2 we can now easily prove

**Theorem 3.2.** Suppose we have a function \( B \) defined in \( O_Q \) satisfying the conditions (3.4), (3.5) above. Then the constant \( C_{s,T} \) in (1.2) is at most \( AQ^2 \).

3.1. Sharpness of \( Q^2 \) in the strong estimate for test functions. We introduce the following function of 3 real variables

\[ B(u, v) := \sup \frac{1}{J} \sum_{J \in D(J)} (\langle fw^{-1}\rangle_{1+} - \langle fw^{-1}\rangle_{1-})^2(w)_J I \]

where supremum is taken over all \( w \in A_2, [w]_{A_2} \leq Q \), such that
\[ \langle w \rangle_J = u, \langle w^{-1} \rangle_J = v. \]

Notice that our function does not depend on \( J \). It is obviously homogeneous \( B(u/t, tv) = t B(u, v) \).

It is easy to check that such \( B \) is defined in \( O_Q \) and satisfies there (3.4) whenever \( (u, v), (u+, v+), (u-, v-) \in O_Q \) and \( (u, v) = \frac{1}{2}((u+, v+) + (u-, v-)) \). Mollifying it we can think that it is smooth. And then in a slightly smaller domain it satisfies (3.1). And we also can keep homegeneity. But then we reduce this to some differential inequality as above on function
\[ \phi \text{ and it is not difficult to see that any solution } \phi \text{ should be of the order } Q^2 \text{ somewhere on } [1, Q]. \text{ This immediately gives us the sharpness.} \]

4. **Strong estimate for all functions**

**Theorem 4.1.** \( C_s \leq AQ^2 \) and this estimate is sharp.

Let us deduce this result from Theorem 3.2. This the occasion of the so-called weighted T1 theorem.

We use the notation \( h_I \) for a standard Haar function supported on dyadic interval \( I \), it is given by

\[
 h_I = \begin{cases} 
 \frac{1}{\sqrt{I}}, & \text{on } I+ \\
 -\frac{1}{\sqrt{I}}, & \text{on } I-. 
\end{cases}
\]

It is an orthonormal basis in unweighted \( L^2 \). Now consider the same type of Haar basis but in weighted \( L^2(w^{-1}) \): functions \( h^I_{w^{-1}} \) are orthogonal to constants in \( L^2(w^{-1}) \), normalized in \( L^2(w^{-1}) \), assume constant value on each child of \( I \), and are supported on \( I \). For dyadic intervals on the line we get

**Lemma 4.2.** The following holds \( h_I = \alpha^I_{w^{-1}} h^I_{w^{-1}} + \beta^I_{w^{-1}} \chi_I \) with

\[
\alpha^I_{w^{-1}} = \frac{\langle w^{-1} \rangle_I^{1/2} \langle w^{-1} \rangle_I^{1/2}}{\langle w^{-1} \rangle_I}, \quad \beta^I_{w^{-1}} = \frac{\langle w^{-1} \rangle_I - \langle w^{-1} \rangle_I^-}{\langle w^{-1} \rangle_I^-}
\]

The sum we want to estimate in (1.5)

\[
\Sigma := \frac{1}{J} \int_J S^2((f w^{-1})(x) w(x)) dx = \frac{1}{J} \sum_{I \in D(J)} (\langle f w^{-1} \rangle_I + \langle f w^{-1} \rangle_I^-)^2 \langle w \rangle_I
\]

is of course

\[
\Sigma = \frac{2}{J} \sum_{I \in D(J)} (f w^{-1}, h_I)^2 \langle w \rangle_I.
\]

We can plug the decomposition of Lemma 4.2 and take ito account that for dyadic lattice \( \alpha^I_{w^{-1}} \leq 2 \langle w^{-1} \rangle_I^{1/2} \). Then we obtain

\[
\Sigma \leq \frac{8}{J} \sum_{I \in D(J)} (f w^{-1}, h^I_{w^{-1}})^2 \langle w \rangle_I \langle w \rangle_I + \frac{2}{J} \sum_{I \in D(J)} \langle f w^{-1} \rangle_I^2 \langle \beta^I_{w^{-1}} \rangle_I^2 \langle w \rangle_I =: \Sigma_1 + \Sigma_2.
\]

The system \( \{h^I_{w^{-1}}\}_{I \in D(J)} \) is orthonormal in \( L^2(w^{-1}) \), and \( \langle w^{-1} \rangle_I \langle w \rangle_I \leq Q \). Hence it is very easy to estimate \( \Sigma_1 \):

\[
\Sigma_1 \leq 8Q \|f\|_{L^2(w^{-1})}^2.
\]

We are left to estimate \( \Sigma_2 \).

To do that let us rewrite \( \Sigma_2 \):

\[
\Sigma_2 = \frac{2}{J} \sum_{I \in D(J)} \left( \frac{\langle f w^{-1} \rangle_I}{\langle w^{-1} \rangle_I} \right)^2 \gamma_I I = \frac{2}{J} \sum_{I \in D(J)} \left( \frac{\langle f w^{-1} \rangle_I}{\langle w^{-1} \rangle_I} \right)^2 \gamma_I I,
\]

where \( \langle \cdot \rangle_{I, w^{-1}} \) means the average with respect to measure \( \mu := w^{-1}(x) dx \) and

\[
\gamma_I := \left( \langle w^{-1} \rangle_I + \langle w^{-1} \rangle_I^- \right)^2 \langle w \rangle_I.
\]
Denote by $M^d_\mu$ the dyadic maximal function with respect to measure $\mu = w^{-1}dx$. Notice then
\begin{equation}
|\langle f w^{-1}\rangle_{I,w^{-1}}| \leq \inf_{x \in I} (M^d_\mu f)(x).
\end{equation}

Here is the lemma that finishes the estimate of $\Sigma_2$ by $A'Q^2 \|f\|_{L^2(w^{-1})}^2$.

**Lemma 4.3.** Let $\mu$ be a positive measure, $\{\tau_I\}_{I \in D}$ be a $\mu$-Carleson sequence with Carleson constant $B$. Let $F$ be any positive function. Then
\[ \sum_{I \in D} (\inf_I F) \cdot \alpha_I \leq B \int_{\mathbb{R}} F d\mu. \]

**Proof.** We can think that only finitely many $\alpha$’s are non-zero, we can also think that $B = 1$. Let us start with the smallest intervals $I$ (with non-zero $\alpha$’s). We choose on each of them a set $E_I$, $\mu(E_I) = \alpha_I$. Then we go to parents of all these intervals. Call them $J$ temporarily. Part of $J$ can be taken by $E_I$, $I \in \text{ch}(J)$, but because $B = 1$ we can easily see that there can be chosen a set $E_J$, disjoint with all $E_I$, $I \in \text{ch}(J)$, and such that $\mu(E_J) = \alpha_J$ (this is Carleson property with constant $\leq 1$ on $J$). After building all these sets $E_J$ we go to parents of $J$ and repeat the construction “bottom-up”. Now
\[ \sum_{L \in D} (\inf_L F) \cdot \alpha_L \leq \sum_{L \in D} \int_{E_L} F(x) d\mu(x), \]
and the disjointness of all sets $E_L$ finishes the proof. $\square$

We are left to use (4.2), and to use Lemma 4.3 with $F := (M^d_\mu f)^2$. In fact, notice that Theorem 3.2 exactly claims that the sequence $\{\gamma_I\}_{I \in D}$ is $\mu$-Carleson, where $\mu = w^{-1}dx$, and its Carleson constant is at most $AQ^2$.

Finally we got
\[ \Sigma_2 \leq A'Q^2 \int (M^d_\mu f)^2 d\mu \leq A'Q^2 \|f\|_{L^2(\mu)}^2 = A'Q^2 \|f\|_{L^2(w^{-1})}^2, \]
where we used the dyadic maximal theorem with arbitrary measure $\mu$. Theorem 4.1 is completely proved. As theorem 3.2 is sharp, we get a sharp estimate.

### 5. A Partial Differential Inequality for full weighted square function

In this section we are going to reduce the harmonic analysis problem of sharp estimate of weak norm of weighted square function to a certain Partial Differential Inequality (PDI) with an obstacle condition in a rather simple subdomain of $\mathbb{R}^3$.

We introduce the following function of 5 real variables (compare with (2.1)):
\begin{equation}
B(F, f, u, v, \lambda) := \sup_{|J|} \frac{1}{|J|} w \left\{ x \in J : \sum_{I \in D(J)} |\Delta_f(\phi w^{-1})|^2 \chi_I(x) > \lambda \right\},
\end{equation}
where supremum is taken over all $w \in A_2, [w]_{A_2} \leq Q$, such that $\langle w \rangle_J = u, \langle w^{-1} \rangle_J = v,$ and over all functions $\phi$ such that $\langle \phi w^{-1} \rangle_J = f, \langle \phi^2 w^{-1} \rangle_J = F$. A disclaimer: the reader should not be confused by notations, but should be warned: of course $F, f, u, v$ stand for numbers, not functions.
Notice that by scaling argument our function does not depend on \( J \) but depends on \( Q = [w]_{A_2} \). Notice also that numbers \( F, f \) and \( u, v \) are not independent, in particular,
\[
f^2 \leq F \cdot v, \quad 1 \leq u \cdot v \leq Q, \quad \lambda \geq 0.
\]
This describes the domain \( \Omega = \Omega_Q \) of definition of function \( B \). Let
\[
\sup_{\Omega_Q} \frac{\lambda B(F, f, u, v, \lambda)}{F} =: \hat{Q}.
\]
By simple one-point-singularity weight we know that
\[
\hat{Q} \geq Q.
\]
The question is of course whether one of the following is true
\[
\begin{cases}
\exists A < \infty \quad \hat{Q} \leq AQ \\
\limsup_{Q \to \infty} \frac{\hat{Q}}{Q} = \infty \\
\exists \varepsilon > 0, a_\varepsilon > 0 : \hat{Q} \geq a_\varepsilon Q (\log Q)^\varepsilon
\end{cases}
\]
From [3], [1] we know (recall that \( \hat{Q} \) is not the weak norm but is the square of the weak norm \( ||S : L^2(w) \to L^{2,\infty}(w)|| \)):
\[
\hat{Q} \leq AQ \log Q.
\]
The goal of this section is to reduce the estimate of \( \hat{Q} \) to a solution of a PDI in a certain simple domain in \( \mathbb{R}^3 \).

**Lemma 5.1.** Function \( B \) introduced above is 1) even in variable \( f \), 2) decreases in variable \( \lambda \), 3) it is not greater than \( u \).

**Proof.** All the claims 1)–3) follow immediately from the definition of \( B \). \( \square \)

**5.1. Homogeneity.** Let \( B_1(F, f, u, v) := B(F, f, u, v, 1) \). From the definition we can write
\[
B(F, f, u, v, \lambda) = \frac{1}{\sqrt{\lambda}} B_1\left(\frac{F}{\sqrt{\lambda}}, \frac{f}{\sqrt{\lambda}}, u\sqrt{\lambda}, \frac{v}{\sqrt{\lambda}}\right).
\]
But also \( B(F, f, u, v, \lambda) = B(s^2F, sf, u, v, s^2\lambda) \), hence
\[
B(F, f, u, v, \lambda) = B_1\left(\frac{F}{\lambda}, \frac{f}{\sqrt{\lambda}}, u, \frac{v}{\sqrt{\lambda}}\right).
\]
Hence \( B_1\left(\frac{F}{\lambda}, \frac{f}{\sqrt{\lambda}}, u, v\right) = \frac{1}{\sqrt{\lambda}} B_1\left(\frac{F}{\sqrt{\lambda}}, \frac{f}{\sqrt{\lambda}}, u\sqrt{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \). If we denote
\[
\alpha := \frac{F}{\sqrt{\lambda}}, \quad \beta := \frac{f}{\sqrt{\lambda}}, \quad \gamma := u\sqrt{\lambda}, \quad \tau := \frac{v}{\sqrt{\lambda}},
\]
then we just obtained
\[
\frac{1}{\sqrt{\lambda}} B_1(\alpha, \beta, \gamma, \tau) = B_1\left(\frac{\alpha}{\sqrt{\lambda}}, \frac{\beta}{\sqrt{\lambda}}, \frac{\gamma}{\sqrt{\lambda}}, \tau\sqrt{\lambda}\right).
\]
Plug in this equality now \( \sqrt{\lambda} = 1/\tau \). Then,
\[
B_1(\alpha, \beta, \gamma, \tau) = \frac{1}{\tau} B_1(\alpha\tau, \beta, \gamma, 1).
\]
Finally let us make one more change of variables:
\[
a := \tau \alpha := \frac{v}{\sqrt{\lambda}} F, \quad b := \beta := \frac{f}{\sqrt{\lambda}}, \quad C := \tau \gamma := \frac{v}{\sqrt{\lambda}} u\sqrt{\lambda} = uv.
\]
Denote
\[ \Xi_Q(a, b, C) := B_1(a, b, C, 1). \]

Then
\[ B_1(\alpha, \beta, \gamma, \tau) = \frac{1}{\tau} \Xi(a, b, C), \] where \( a = \alpha \tau, b = \beta, C = \gamma \tau. \)

By (5.7) and (5.2) the domain of definition of \( \Xi := \Xi_Q \) is
\[ O := O_Q := \{(a, b, C) \in \mathbb{R}^3 : b^2 \leq a, 1 \leq C \leq Q\}. \]

In particular, \( \hat{Q} \) from (5.3) can be now defined as
\[ \hat{Q} = \sup_{O_Q} \frac{\Xi(a, b, C)}{a}. \]

5.2. The main inequality and the Partial Differential version of it. Exactly like in our first section we can write the following main inequality. If \( P = (F, f, u, v, \lambda), P_+ = (F+, f+, u+, v+, \lambda+), P_- = (F-, f-, u-, v-, \lambda-) \) belong to \( \Omega_Q \), and \( F = \frac{1}{2}(F + F-) \), \( f = \frac{1}{2}(f + f-) \), \( u = \frac{1}{2}(u + u-) \), \( v = \frac{1}{2}(v + v-) \), \( \lambda = \min(\lambda+, \lambda-) \), then the main inequality holds with constant \( \kappa = 1 \):
\[ (5.11) \quad B(F, f, u, v, \lambda + \kappa(f + -f -)^2) - \frac{1}{2}(B(P +) + B(P -)) \geq 0. \]

In infinitesimal form this inequality becomes
\[ (5.12) \quad \frac{1}{2} d_{F,f,u,v}^2 B - \kappa \frac{\partial B}{\partial \lambda} \cdot (df)^2 \leq 0. \]

Remark. Notice that by definition \( B \) decreases in \( \lambda \), so this inequality is some qualified concavity of \( B \) in variables \((F, f, u, v)\) gauged by negative \( \frac{\partial B}{\partial \lambda} \).

This means that a certain matrix of second and first variables of \( B = B_Q \) of size \( 4 \times 4 \) is negatively defined at every point \((F, f, u, v, \lambda)\) in the interior of the domain \( \Omega = \Omega_Q \) defined in (5.2).

However we changed the variables and reduced everything to \( \Xi = \Xi_Q \) of 3 variables in \( O := O_Q \) from (5.9).

After direct calculation one can reduce (5.12) to the following PDI on the second differential form of \( \Xi \) (with drift=first order terms) in \( O_Q \):
\[ (5.13) \quad \begin{bmatrix} \Xi_{aa} & \Xi_{ab} & \Xi_{aC} & 0 \\ \Xi_{ab} & \Xi_{bb} + \kappa(b \Xi_b + 2a \Xi_a) & \Xi_{bC} & -\Xi_b \\ \Xi_{aC} & \Xi_{bC} & \Xi_{CC} & 0 \\ 0 & -\Xi_b & 0 & 2\Xi - 2a \Xi_a - 2C \Xi_C \end{bmatrix} \leq 0 \]
in \( O_Q \).

5.3. Obstacle condition. This Partial Differential Inequality should be coupled with the obstacle condition exactly in the same manner as in the first section.

By the same reasoning as in the first section we can conclude that for all large \( Q \) and all point \( P = (F, f, u, v, \lambda) \in \Omega_Q \) such that \( uv \geq 10 \) one would have the following condition on \( B = B_Q \) (called the obstacle condition):
\[ (5.14) \quad B(P) = u, \] if \( f^2 \geq b_0 \lambda \) for a certain positive absolute \( b_0 \).

In terms of \( \Xi_Q \) this can be immediately rewritten as the condition on points in \( O = O_Q \)
(5.15) \( \Xi(a, b, C) = C \), if \( b \geq b_0 \) and \( C \geq 10 \) for a certain positive absolute \( b_0 \).

**Theorem 5.2.** Suppose that \( \limsup_{Q \to \infty} \hat{Q}/Q = \infty \), where \( \hat{Q} := \sup_{O_Q} \frac{\Xi_Q(a, b, C)}{a} \), where \( O_Q := \{(a, b, C) : b^2 \leq a, 1 \leq C \leq Q \} \) and \( \Xi_Q \) satisfies the obstacle condition (5.15) and concavity condition (5.13) for \( \kappa = 1 \). Then weak weighted norm inequality of dyadic square function cannot be at most constant times \( Q^{1/2} \) for all large \( Q \), where \( Q = [w]_{A_2} \). In other words, then the blow-up happens.

The converse is also true in a sense. Let us write a stronger obstacle condition:

(5.16) \( \Xi(a, b, C) = C \), if \( b \geq b_0 \) for a certain positive absolute \( b_0 \).

**Remark.** Suppose that \( \limsup_{Q \to \infty} \hat{Q}/Q < \infty \), where \( \hat{Q} := \sup_{O_Q} \frac{\Xi_Q(a, b, C)}{a} \), where \( O_Q := \{(a, b, C) : b^2 \leq a, 1 \leq C \leq Q \} \) and \( \Xi_Q \) satisfies a stronger obstacle condition (5.16) and concavity condition (5.13) for \( \kappa > 0 \). Then weak weighted norm inequality of dyadic square function is believed to be at most constant times \( Q^{1/2} \), where \( Q = [w]_{A_2} \). Why we write “believed to be” instead of “will be”? Because the fact that \( \Xi_Q \) exists and satisfies (5.16) and concavity condition (5.13) for \( \kappa > 0 \) gives only the infinitesimal version of the main inequality. To deduce the estimate of square function we need non-infinitesimal version of the main inequality. In the first section such passage from one version to another was successful, but now we have a more involved situation.

5.4. Rescaling and blow-up to eliminate \( Q \) from the questions. Let us consider our functions \( \Xi_Q \) in \( O_Q \) and lets us make one more change of variables. This time it is a simple blow-up. Namely, let us consider

\[ \eta_Q := \frac{1}{Q} \Xi_Q(a, b, cQ) \]

in the domain \( G_Q := \{(a, b, c) : b^2 \leq a, \frac{1}{Q} \leq c \leq 1 \} \). It is easy to see that the form (5.13) is invariant under this change. If

\[ \limsup_{Q \to \infty} \frac{\hat{Q}}{Q} < \infty \]

were true, we would be able to tend \( Q \) to infinity, and, noticing, that (5.13) applied \( \eta_Q \) will be preserved, we come to the following question in the limiting function \( \eta \) in limit domain

(5.17) \( G := \{(a, b, c) : b^2 \leq a, 0 < c \leq 1 \} \)

with the limiting obstacle condition:

(5.18) \( \eta(a, b, c) = c \), if \( b \geq b_0 \) for a certain positive absolute \( b_0 \).

So the question is: can one have a function \( \eta \) in the domain \( G := \{(a, b, c) : b^2 \leq a, 0 < c \leq 1 \} \) satisfying the size condition

(5.19) \( \eta(a, b, c) \leq \text{Const} \cdot a \),

the obstacle condition (5.18) and satisfying the following PDI in \( G \):

\[
\begin{bmatrix}
\eta_{aa} & \eta_{ab} & \eta_{ac} & 0 \\
\eta_{ba} & \eta_{bb} + \kappa(b \eta_b + 2a \eta_a) & \eta_{bc} & -\eta_b \\
\eta_{ca} & \eta_{cb} & \eta_{cc} & 0 \\
0 & -\eta_b & 0 & 2\eta - 2a \eta_a - 2c \eta_c
\end{bmatrix} \leq 0
\]
We believe that any function $\eta(a, b, c)$ solving (5.20) (with $\kappa > 0$) and (5.18) should grow much faster than $a$ in the sense that $\eta/a$ cannot be bounded in $G$. We actually already proved above

**Theorem 5.3.** Suppose there is no function $\eta$ even in $b$ satisfying (5.19), (5.20), (5.18) and $2a\eta_a + b\eta_b \geq 0$ in the domain $G$, then the blow-up happens.

5.5. **Observation:** By Lemma 5.1 function $B(F, f, u, v, \lambda)$ is decreasing in $\lambda$, so this implies (after all our change of variables and a renormalization) that $2a\eta_a + b\eta_b \geq 0$. This together with negative definiteness of the above matrix implies that $\eta$ is a concave function. Concavity of $\eta$ and the fact that $\eta(a, b, c) = c$ for $b \geq b_0$ gives $\eta(a, b, c) = c$ in $G \cap \{a \geq b_0^2\}$. As $\eta \leq c$ (this easily follows from Lemma 5.1) we have $\eta(a, b, 0) = 0$. Since $(4, 4)$ entry of the matrix is nonpositive, this implies that concave function $\varphi(t) := \eta(at, 0, ct)$, $\varphi(0) = 0$ satisfies the inequality $\varphi(t) - t\varphi'(t) \leq 0$. This can happen if and only if $\varphi(t) = Lt$ for some constant $L \geq 0$. Using concavity of $\eta$ and its obstacle condition one more time we obtain $\eta(a, b, c) = c$ whenever $c \leq \frac{a - b_0^2}{b_0^2 - b^2}$, $a \geq b^2$ and $b^2 \leq b_0^2$.

Notice that also by Lemma 5.1 we are looking for function $\eta$ even in $b$.

Let us look at $\eta$ on the boundary $D := \{a = b^2, 0 < c \leq 1\}$. In the first section we built a function $\Theta(\gamma, \tau) = \min(\gamma, Q \int_0^\tau e^{s^2/2} ds)$. We made lots of changes of variables and a renormalization, but one can follow them. Then one may try the following boundary values of $\eta$ on $D$:

$$\xi(b, c) := \eta(b^2, b, c) = \min(c, b \int_0^b e^{s^2/2} ds).$$

Of course if we build $\eta$ with such boundary condition, satisfying (5.20) and $2a\eta_a + b\eta_b \geq 0$ and such that

$$\eta(a, b, c) \leq Const \cdot a$$

in $G = \{a \geq b^2, 0 < c \leq 1\}$, this will be a strong indication that there is no blow-up in the full weak norm weighted square function estimate. But still this will not be a proof because we need to pass from $\eta$ (the renormalized function of $\Xi_Q$) to $\Xi_Q$ functions themselves and on the top of that transfer the PDI (the main inequality in infinitesimal form) for them to the real non-infinitesimal main inequality.

On the other hand if we prove that no $\eta$ satisfying (5.20), (5.18) and $2a\eta_a + b\eta_b \geq 0$ and such that

$$\eta(a, b, c) \leq Const \cdot a$$

in $G = \{a \geq b^2, 0 < c \leq 1\}$ exists, then we prove that blow-up definitely happens in the full weak norm weighted square function estimate.

**References**


Department of Mathematics, Michigan State University, East Lansing, MI. 48823
*E-mail address: ivanisvi@math.msu.edu* (P. Ivanisvili)

Department of Mathematics, Kent State University, USA
*E-mail address: nazarov@math.kent.edu* (F. Nazarov)

Department of Mathematics, Michigan State University, East Lansing, MI. 48823
*E-mail address: volberg@math.msu.edu* (A. Volberg)