

ON A TWO-PHASE PROBLEM FOR HARMONIC MEASURE IN GENERAL DOMAINS

JONAS AZZAM, MIHALIS MOURGOLOU, XAVIER TOLSA, AND ALEXANDER VOLBERG

ABSTRACT. We show that, for disjoint domains in the Euclidean space, mutual absolute continuity of their harmonic measures implies absolute continuity with respect to surface measure and rectifiability in the intersection of their boundaries. This improves on our previous result which assumed that the boundaries satisfied the capacity density condition.

CONTENTS

1. Introduction	1
2. Harmonic Measure Preliminaries	2
3. The Alt-Caffarelli-Friedman monotonicity formula	4
4. Tangent measures	6
5. The blowup lemmas	7
6. Riesz transforms	12
References	17

1. INTRODUCTION

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be disjoint domains, and let $E \subset \partial\Omega_1 \cap \partial\Omega_2$. In 1990 C. Bishop [Bi1] conjectured that if the respective harmonic measures of Ω_1 and Ω_2 are mutually absolutely continuous, then they should be also mutually absolutely continuous with respect to the n -dimensional Hausdorff measure on an n -rectifiable set \mathbb{R}^{n+1} . In the work [AMT] we proved this conjecture under the assumption that Ω_1 and Ω_2 satisfy the so called capacity density condition (CDC) (although we obtained the stronger property that the set of tangent points for $\partial\Omega_1$ has positive \mathcal{H}^n -measure). In the present paper we prove this in full generality. The precise result reads as follows.

Theorem 1.1. *For $n \geq 2$, let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be two domains and denote by ω_1 and ω_2 their respective harmonic measures. Let $E \subset \partial\Omega_1 \cap \partial\Omega_2$ be a Borel set such that $\omega_1|_E \ll \omega_2|_E \ll \omega_1|_E$. Then E contains an n -rectifiable subset F with $\omega_1(E \setminus F) = 0$ such that $\omega_1|_F$ and $\omega_2|_F$ are mutually absolutely continuous with respect to $\mathcal{H}^n|_F$.*

We remark that, in the planar case $n = 1$, the analogous conclusion had been proved previously by Bishop in [Bi2]. Another partial result was obtained by Kenig, Preiss and Toro in [KPT]. Therein, they showed, among others, that if Ω_1 and $\Omega_2 = \text{ext}(\Omega_1)$ are NTA domains

2010 *Mathematics Subject Classification.* 31A15, 28A75, 28A78.

The first three authors were supported by the ERC grant 320501 of the European Research Council (FP7/2007-2013). X.T. was also supported by 2014-SGR-75 (Catalonia), MTM2013-44304-P (Spain), and by the Marie Curie ITN MAnET (FP7-607647). A.V. was supported by NSF Grant DMS1600065.

with mutually absolutely continuous harmonic measures, then these harmonic measures are concentrated on a set of dimension n .

As in [AMT], the main tools to prove the preceding result stated in Theorem 1.1 are:

- a blowup argument for harmonic measure inspired by the techniques from Kenig, Preiss and Toro [KPT],
- the Alt-Caffarelli-Friedman monotonicity formula [ACF], and
- a rectifiability criterion by Girela-Sarrión and Tolosa [GT], which in turn uses techniques which arise from the solution of the David-Semmes problem by Nazarov, Tolosa and Volberg [NTV1], [NTV2] and the work of Eiderman, Nazarov, and Volberg [ENV].

The main new tool that we use in the present paper is a new blowup argument which does not require the CDC and yields the local convergence in L^2 of some subsequences of rescaled Green functions. This technique was used very recently in [TV] to obtain a new proof of Tsirelson's theorem [Ts]. Instead, the blow argument in [AMT] yields local uniform convergence of the rescaled Green functions.

2. HARMONIC MEASURE PRELIMINARIES

We will need the following classical result (see [AHM³TV], for example):

Lemma 2.1. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Denote by ω^p its harmonic measure with pole at $p \in \Omega$ and by G its Green function. Let $B = B(x_0, r)$ be a closed ball with $x_0 \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$. Then, for all $a > 0$,*

$$(2.1) \quad \omega^x(aB) \gtrsim \inf_{z \in 2B \cap \Omega} \omega^z(aB) r^{n-1} G(x, y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega,$$

with the implicit constant independent of a .

The next lemma is usually known as Bourgain's estimate. See [AHM³TV] for the precise formulation below.

Lemma 2.2. *There is $\delta_0 > 0$ depending only on $n \geq 1$ so that the following holds for $\delta \in (0, \delta_0]$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, $n - 1 < s \leq n + 1$, $\xi \in \partial\Omega$, $r > 0$, and $B = B(\xi, r)$. Then*

$$\omega^x(B) \gtrsim_{n,s} \frac{\mathcal{H}_\infty^s(\delta B \setminus \Omega)}{(\delta r)^s} \quad \text{for all } x \in \delta B \cap \Omega.$$

In the the next lemma we reduce the proof of Theorem 1.1 to the case when the domains Ω_1, Ω_2 are Wiener regular. The proof is from [HMMTV], but as this paper will not be published, we recreate the details here with some slight modifications.

Lemma 2.3. *Let Ω_1 and Ω_2 be two disjoint connected domains in \mathbb{R}^{n+1} with harmonic measures $\omega_i = \omega_{\Omega_i}^{p_i}$ for some $p_i \in \Omega_i$ and suppose $\omega_1 \ll \omega_2 \ll \omega_1$ on a Borel set $E \subset \partial\Omega_1 \cap \partial\Omega_2$. Then there are Wiener regular subdomains $\tilde{\Omega}_i \subset \Omega_i$ containing p_i for $i = 1, 2$ and $G_0 \subset E$ with $\tilde{\omega}_i(G_0) > 0$ upon which $\tilde{\omega}_2 \ll \tilde{\omega}_1 \ll \tilde{\omega}_2$.*

Proof. Let F_i be the irregular points for Ω_i . By [AG, Theorem 6.6.8] these sets are polar sets in \mathbb{R}^{n+1} . By [H, Lemma 6.4.6], there is a positive superharmonic function v_i on Ω_i so

that

$$\lim_{\substack{y \rightarrow x \\ y \in \Omega_i}} v_i(y) = \infty \text{ for all } x \in F := F_1 \cup F_2.$$

Let $\lambda > 0$. Since v_i is superharmonic on Ω_i , it is lower semicontinuous, and remains so when we extend it by zero to Ω_i^c . Thus, for each $x \in F$ there is a closed ball $B_i(x)$ centered at x (not containing either p_i) such that $v_i \geq \lambda$ on $B_i(x) \cap \Omega_i$. Let $\{B_j\}$ be a Besicovitch subcovering. Let $H = \bigcup B_j$ and

$$\tilde{\Omega}_i = \Omega_i \setminus H.$$

Note that $\tilde{\Omega}_i$ is open. Indeed, to show that $\tilde{\Omega}_i^c$ is closed consider $x_k \in \tilde{\Omega}_i^c$, $k \geq 1$ and $x_k \rightarrow x$. Then we need to show $x \in \tilde{\Omega}_i^c$. If there is a subsequence contained in Ω_i^c , we are done. Otherwise, assume that $x_k \in H \setminus \Omega_i^c = H \cap \Omega_i$. If $x_k \in B_j$ for infinitely many k , then $x \in B_j$ and we are done since B_j is closed and $B_j \subset \tilde{\Omega}_i^c$. Otherwise, suppose x_k is not in any B_j more than finitely many times. By the bounded overlap property, if $j(x_k)$ is such that $x_k \in B_{j(x_k)}$, then $r(B_{j(x_k)}) \downarrow 0$ as $k \rightarrow \infty$, and since the balls are centered on $F \subset \Omega_i^c$, $x \in \Omega_i^c \subset \tilde{\Omega}_i^c$, and we are done. Thus, $\tilde{\Omega}_i$ is open.

Let $\tilde{\omega}_i = \omega_{\tilde{\Omega}_i}^{p_i}$. Note that $\tilde{\Omega}_i$ is now a regular domain. Indeed, one need only observe that whenever $\Omega \subset \Omega'$ are two domains and $x \in \partial\Omega \cap \partial\Omega'$ is regular for Ω' , then it is regular for Ω . Hence, if $x \in \partial\tilde{\Omega}_i$, then either $x \in \partial B_i$ for some i , in which case x is regular for $B_i^c \supset \tilde{\Omega}_i$, or $x \in \partial\Omega_i \setminus F$, in which case x is regular for $\Omega_i \supset \tilde{\Omega}_i$ since it is not in F , and either case implies x is regular for $\tilde{\Omega}_i$.

Let $G = E \setminus H$. By the maximum principle on $\tilde{\Omega}_i$, since $u/\lambda \geq 1$ on H , $\omega_i(H) \leq \lambda v_i(p_i)$. Picking

$$\lambda < \frac{1}{2} \min\{\omega_i(E)/v_i(p_i)\},$$

this gives $\omega_i(H) \leq \frac{1}{2}\omega_i(E)$ and hence $\omega_i(G) > 0$. Similarly, by the maximum principle, since $u/\lambda \geq 1$ on H , $\tilde{\omega}_i(H) \leq \lambda v_i(p_i)$. Picking

$$\lambda < \frac{1}{2} \min\{\omega_i(G)/v_i(p_i)\},$$

this gives

$$\tilde{\omega}_i(H) \leq \frac{1}{2}\omega_i(G).$$

Moreover, by the maximum principle, and since $\tilde{\Omega}_i$ is a regular domain,

$$\tilde{\omega}_i(H^c \cap G^c) \leq \omega_i(H^c \cap G^c).$$

Thus,

$$\begin{aligned} \tilde{\omega}_i(G) &= 1 - \tilde{\omega}_i(G^c) = 1 - \tilde{\omega}_i(H \cap G^c) - \tilde{\omega}_i(H^c \cap G^c) \\ &\geq 1 - \frac{1}{2}\omega_i(G) - \omega_i(H^c \cap G^c) \geq \omega_i(G) - \frac{1}{2}\omega_i(G) = \frac{1}{2}\omega_i(G) > 0 \end{aligned}$$

Note that $\tilde{\omega}_1 \ll \omega_1$ on G by the maximum principle (or by Carleman's principle, see [HKM, Theorem 11.3(b)]), and since $\tilde{\omega}_1(G) > 0$, it is not hard to show using the Lebesgue decomposition theorem that there is $G_1 \subset G$ of full $\tilde{\omega}_1$ -measure upon which we also have $\omega_1 \ll \tilde{\omega}_1$. Hence $\omega_1(G_1) > 0$, which implies $\omega_2(G_1) > 0$. The same reasoning gives us a set $G_2 \subset G_1$ upon which $\tilde{\omega}_2 \ll \omega_2 \ll \tilde{\omega}_2$. Thus, $\tilde{\omega}_2 \ll \tilde{\omega}_1 \ll \tilde{\omega}_2$ on G_2 . \square

Remark 2.4. *In light of this lemma, for the remainder of this paper, to prove Theorem 1.1 we shall assume our domains Ω_1, Ω_2 are Wiener regular.*

3. THE ALT-CAFFARELLI-FRIEDMAN MONOTONICITY FORMULA

The following theorem contains the Alt-Caffarelli-Friedman monotonicity formula:

Theorem 3.1. [CS, Theorem 12.3] *Let $B(x, R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x, R)) \cap C(B(x, R))$ be nonnegative subharmonic functions. Suppose that $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Set*

$$(3.1) \quad \gamma(x, r) = \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}} dy \right) \cdot \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}} dy \right).$$

Then $\gamma(x, r)$ is a non-decreasing function of $r \in (0, R)$ and $\gamma(x, r) < \infty$ for all $r \in (0, R)$. That is,

$$(3.2) \quad \gamma(x, r_1) \leq \gamma(x, r_2) < \infty \quad \text{for } 0 < r_1 \leq r_2 < R.$$

We remark that the preceding result was also stated in [AMT], although under somewhat stronger assumptions. In the current paper we will apply the preceding formula to the case when Ω_1 and Ω_2 are disjoint Wiener regular domains, $x \in \partial\Omega_1 \cap \partial\Omega_2$, with u_1, u_2 equal to the Green functions of Ω_1, Ω_2 with poles at p_1, p_2 , extended by 0 to Ω_1^c, Ω_2^c . In this case, it is well known that $u_i \in W_{loc}^{1,2}(\mathbb{R}^{n+1} \setminus \{p_i\}) \cap C(\mathbb{R}^{n+1} \setminus \{p_i\})$ for $i = 1, 2$ and so the assumptions of the preceding theorem are satisfied in any ball which does contain p_1 and p_2 .

Arguing as in [KPT, Theorem 3.3], we obtain:

Lemma 3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be as in Theorem 1.1, and assume further that they are Wiener regular. For $i = 1, 2$, let ω_i be the harmonic measure of Ω_i with pole at $p_i \in \Omega_i$. Let $0 < R < \min_i \text{dist}(p_i, \partial\Omega_i)$. Then for $0 < r < R/4$ and $\xi \in \partial\Omega_1 \cap \partial\Omega_2$,*

$$(3.3) \quad \frac{\omega_i(B(\xi, r))}{r^n} \lesssim \left(\frac{1}{r^2} \int_{B(\xi, 2r)} \frac{|\nabla u_i(y)|^2}{|y-\xi|^{n-1}} dy \right)^{\frac{1}{2}} \lesssim \left(\frac{1}{r^{n+3}} \int_{B(\xi, 4r)} |u_i|^2 \right)^{\frac{1}{2}}$$

and in particular,

$$(3.4) \quad \frac{\omega_1(B(\xi, r))}{r^n} \frac{\omega_2(B(\xi, r))}{r^n} \lesssim \gamma(\xi, 2r)^{\frac{1}{2}},$$

where $\gamma(\xi, 2r)$ is defined by (3.1).

Lemma 3.3. *Let $\Omega_1 \subset \mathbb{R}^{n+1}$ be a Wiener regular domain and denote by ω_1 its harmonic measure with pole at $p_1 \in \Omega_1$. Let B be a ball centered at $\partial\Omega_1$ such that $p_1 \notin 10B$. Suppose that $\omega_1(4B) \leq C \omega_1(\delta_0 B)$ and $\mathcal{H}^{n+1}(B \setminus \Omega_1) \geq C^{-1} r(B)^{n+1}$. Then,*

$$\mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \gtrsim r(B)^{n+1}.$$

Proof. Let φ be a non-negative bump function which equals 1 on $\delta_0 B$ and is supported on $2\delta_0 B$. Then we have:

$$\omega_1(\delta_0 B) \leq \int \varphi d\omega_1 = \int \Delta\varphi(y) u_1(y) dy \lesssim \frac{1}{r^2} \mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \sup_{y \in 2\delta_0 B} u_1(y),$$

where u_1 is the Green function with pole at p_1 . From Bourgain's estimate (taking into account that $\mathcal{H}^{n+1}(B \setminus \Omega_1) \geq C^{-1}r(B)^{n+1}$) and Lemma 2.1 we deduce that

$$\sup_{y \in 2\delta_0 B} u_1(y) \lesssim \frac{\omega_1(4B)}{r^{n-1}},$$

and so

$$\omega_1(\delta_0 B) \lesssim \frac{1}{r^{n+1}} \mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \omega_1(4B).$$

Using then that $\omega_1(4B) \leq C \omega(\delta_0 B)$, the lemma follows. \square

Lemma 3.4. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be as in Theorem 1.1, and assume further that they are Wiener regular. For $i = 1, 2$, let ω_i be the harmonic measure of Ω_i with pole at $p_i \in \Omega_i$. Let B be a ball centered at $\partial\Omega_1 \cap \partial\Omega_2$ such that $p_1, p_2 \notin 10B$. Suppose that $\omega_i(4B) \leq C \omega_i(\delta_0 B)$ for $i = 1, 2$. Then,*

$$\mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_1) \approx \mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_2) \approx r(B)^{n+1}$$

and

$$\sup_{y \in \delta_0 B} u_i(y) \lesssim \frac{\omega_i(4B)}{r^{n-1}} \quad \text{for } i = 1, 2.$$

Proof. There exists $1 \leq i \leq 2$ such that

$$(3.5) \quad \mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_i) \geq C^{-1}r(B)^{n+1}.$$

Suppose this holds for Ω_1 , then we only need to show now that $\mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_2) \geq C^{-1}r(B)^{n+1}$. Since $\delta_0 < 1/2$, $B \supset 2\delta_0 B$, and hence (3.5) implies $\mathcal{H}^{n+1}(B \setminus \Omega_1) \geq C^{-1}r(B)^{n+1}$. From Lemma 3.3, we infer that

$$\mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_2) \geq \mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \gtrsim r(B)^{n+1}.$$

The second statement in the lemma follows from these estimates in combination with Bourgain's estimate and Lemma 2.1. \square

Lemma 3.5. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be as in Theorem 1.1, and assume further that they are Wiener regular. For $i = 1, 2$, let ω_i be the harmonic measure of Ω_i with pole at $p_i \in \Omega_i$. Let $0 < R < \min_i \text{dist}(p_i, \partial\Omega_i)$. Let $\xi \in \partial\Omega_1 \cap \partial\Omega_2$ and $r < \delta_0 R/4$, $i = 1, 2$. Suppose that $\omega_i(B(\xi, 4r)) \leq C \omega_i(B(\xi, \delta_0 r))$ for $i = 1, 2$. Then we have*

$$(3.6) \quad \left(\frac{1}{r^{n+1}} \int_{B(\xi, r) \cap \Omega_i} |\nabla u_i|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{1}{r^{n+3}} \int_{B(\xi, 2r) \cap \Omega_i} |u_i|^2 \right)^{\frac{1}{2}} \lesssim \frac{\omega_i(B(\xi, 8\delta_0^{-1}r))}{r^n}.$$

In particular,

$$(3.7) \quad \gamma(\xi, r)^{\frac{1}{2}} \lesssim \frac{\omega_1(B(\xi, 8\delta_0^{-1}r))}{r^n} \frac{\omega_2(B(\xi, 8\delta_0^{-1}r))}{r^n},$$

where $\gamma(\xi, r)$ is defined by (3.1).

Proof. Since u_i vanishes continuously at the boundary of $\partial\Omega_i$, we may extend it by zero in $\mathbb{R}^{n+1} \setminus \Omega_i$. Then, as the extended function (which we still denote by u_i) is non-negative and subharmonic in \mathbb{R}^{n+1} , by Caccioppoli's inequality (which still holds for subharmonic functions) and Lemma 3.4, we infer that

$$\left(\int_{B(\xi, r)} |\nabla u_i|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{1}{r^2} \int_{B(\xi, 2r)} (u_i)^2 \right)^{\frac{1}{2}} \lesssim \omega_i(B(\xi, 8\delta_0^{-1}r)) r^{\frac{1-n}{2}}.$$

This shows (3.6), which in turn implies (3.7). \square

4. TANGENT MEASURES

For $a \in \mathbb{R}^{n+1}$ and $r > 0$, we consider the map

$$T_{a,r}(x) = \frac{x - a}{r}.$$

Note that

$$T_{a,r}(B(a, r)) = \mathbb{B} := B(0, 1).$$

Recall also that, given a Radon measure μ , the notation $T_{a,r}[\mu]$ stands for the image measure of μ by $T_{a,r}$. That is,

$$T_{a,r}[\mu](A) = \mu(rA + a), \quad A \subset \mathbb{R}^{n+1}.$$

Given two Radon measure μ and σ , we set

$$F_B(\mu, \sigma) = \sup_f \int f d(\mu - \sigma),$$

where the supremum is taken over all the 1-Lipschitz functions supported on B . For $r > 0$, we write

$$F_r(\mu, \nu) = F_{\overline{B(0,r)}}(\mu, \nu), \quad F_r(\mu) = F_r(\mu, 0) = \int (r - |z|)_+ d\mu.$$

Definition 4.1. [Pr, Section 2]

- (a) A set \mathcal{M} of non-zero Radon measures in \mathbb{R}^{n+1} is a *cone* if $c\mu \in \mathcal{M}$ whenever $\mu \in \mathcal{M}$ and $c > 0$.
- (b) A cone \mathcal{M} is a *d-cone* if $T_{0,r}[\mu] \in \mathcal{M}$ for all $\mu \in \mathcal{M}$ and $r > 0$.
- (c) The *basis* of a *d-cone* \mathcal{M} is the set $\{\mu \in \mathcal{M} : F_1(\mu) = 1\}$.
- (d) For a *d-cone* \mathcal{M} , $r > 0$, and μ a Radon measure with $0 < F_r(\mu) < \infty$, we define the *distance* between μ and \mathcal{M} as

$$d_r(\mu, \mathcal{M}) = \inf \left\{ F_r \left(\frac{\mu}{F_r(\mu)}, \nu \right) : \nu \in \mathcal{M}, F_r(\nu) = 1 \right\}$$

Lemma 4.2 ([KPT] Section 2). *Let μ, ν be Radon measures in \mathbb{R}^{n+1} and \mathcal{M} a *d-cone*. For $\xi \in \mathbb{R}^{n+1}$ and $r > 0$,*

- (1) $T_{\xi,r}[\mu](B(0, s)) = \mu(B(\xi, sr))$,
- (2) $\int f dT_{\xi,r}[\mu] = \int f \circ T_{\xi,r} d\mu$,
- (3) $F_{B(\xi,r)}(\mu) = rF_1(T_{\xi,r}[\mu])$,
- (4) $F_{B(\xi,r)}(\mu, \nu) = rF_1(T_{\xi,r}[\mu], T_{\xi,r}[\nu])$,
- (5) $\mu_i \rightarrow \mu$ weakly if and only if $F_r(\mu_i, \mu) \rightarrow 0$ for all $r > 0$,

- (6) $d_r(\mu, \mathcal{M}) \leq 1$,
- (7) $d_r(\mu, \mathcal{M}) = d_1(T_{0,r}[\mu], \mathcal{M})$,
- (8) if $\mu_i \rightarrow \mu$ and $F_r(\mu) > 0$, then $d_r(\mu_i, \mathcal{M}) \rightarrow d_r(\mu, \mathcal{M})$.

Theorem 4.3 ([Pr] Corollary 2.7). *Let μ be a Radon measure on \mathbb{R}^{n+1} , and $\xi \in \text{supp } \mu$. Then $\text{Tan}(\mu, \xi)$ has compact basis if and only if*

$$(4.1) \quad \limsup_{r \rightarrow 0} \frac{\mu(B(\xi, 2r))}{\mu(B(\xi, r))} < \infty.$$

In this case, $0 \in \text{supp } \nu$ for all $\nu \in \text{Tan}(\mu, \xi)$, and

$$\frac{\nu(B(0, 2r))}{\nu(B(0, r))} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(\xi, 2r))}{\mu(B(\xi, r))} \quad \text{for all } r > 0.$$

Lemma 4.4. [Ma, Lemma 14.6] *Let μ be a Radon measure on \mathbb{R}^n , ϕ a non-negative locally integrable function on \mathbb{R}^{n+1} , and λ the Radon measure such that $\lambda(B) = \int_B \phi d\mu$ for all Borel sets B . Then $\text{Tan}(\mu, x) = \text{Tan}(\lambda, x)$ for λ -almost all $x \in \mathbb{R}^{n+1}$.*

Lemma 4.5. [Ma, Theorem 14.3] *Let μ be a Radon measure on \mathbb{R}^{n+1} . If $\xi \in \mathbb{R}^{n+1}$ and (4.1) holds, then every sequence $r_i \downarrow 0$ contains a subsequence such that $T_{\xi, r_j} \# \mu / \mu(B(\xi, r_j))$ converges to a measure $\nu \in \text{Tan}(\mu, \xi)$.*

Theorem 4.6. [Ma, Theorem 14.16] *Let μ be a Radon measure on \mathbb{R}^{n+1} . For μ -almost every $x \in \mathbb{R}^{n+1}$, if $\nu \in \text{Tan}(\mu, x)$, the following hold:*

- (1) $T_{y,r}[\nu] \in \text{Tan}(\mu, x)$ for all $y \in \text{supp } \nu$ and $r > 0$.
- (2) $\text{Tan}(\nu, y) \subset \text{Tan}(\mu, x)$ for all $y \in \text{supp } \nu$.

5. THE BLOWUP LEMMAS

For a measure μ , $\xi \in \text{supp } \mu$, L an n -plane, and $r > 0$, we define

$$\beta_{\mu,1}^L(\xi, r) = \frac{1}{r^n} \int_{B(\xi, r)} \frac{\text{dist}(x, L)}{r} d\mu(x)$$

and

$$\beta_{\mu,1}(\xi, r) = \inf_L \beta_{\mu,1}^L(\xi, r)$$

where the infimum is over all n -dimensional planes L .

The aim of this section is to prove the following lemma. The proof is a variation on the work in [TV], which in turn is inspired by previous blowup arguments in [AMT] and [KPT].

Lemma 5.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be disjoint domains and suppose there is $E \subset \partial\Omega_1 \cap \partial\Omega_2$ upon which we have $\omega_1|_E \ll \omega_2|_E \ll \omega_1|_E$. Fix $\varepsilon < 1/100$ and let E_m be the set of $\xi \in E$ such that for all $0 < r < 1/m$ and $i = 1, 2$ we have*

$$(5.1) \quad \omega_i(B(\xi, 2r)) \leq m \omega_i(B(\xi, r)),$$

$$(5.2) \quad \mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i) \geq \frac{1}{m} r^{n+1},$$

and

$$(5.3) \quad \beta_{\omega_1,1}(\xi, r) < \varepsilon \frac{\omega_1(\xi, r)}{r^n}$$

Then

$$(5.4) \quad \omega_1 \left(E \setminus \bigcup_{m \geq 1} E_m \right) = 0.$$

Set

$$E^* = \left\{ \xi \in E : \lim_{r \rightarrow 0} \frac{\omega_1(E \cap B(\xi, r))}{\omega_1(B(\xi, r))} = \lim_{r \rightarrow 0} \frac{\omega_2(E \cap B(\xi, r))}{\omega_2(B(\xi, r))} = 1 \right\}.$$

By [Ma, Corollary 2.14 (1)] and because ω_1 and ω_2 are mutually absolutely continuous on E ,

$$\omega_1(E \setminus E^*) = \omega_2(E \setminus E^*) = 0.$$

Also, set

$$\Lambda_1 = \left\{ \xi \in E^* : 0 < h(\xi) := \frac{d\omega_2}{d\omega_1}(\xi) = \lim_{r \rightarrow 0} \frac{\omega_2(B(\xi, r))}{\omega_1(B(\xi, r))} = \lim_{r \rightarrow 0} \frac{\omega_2(E \cap B(\xi, r))}{\omega_1(E \cap B(\xi, r))} < \infty \right\}$$

and

$$\Gamma = \{ \xi \in \Lambda_1 : \xi \text{ is a Lebesgue point for } h \text{ with respect to } \omega_1 \}.$$

Again, by Lebesgue differentiation for measures (see [Ma, Corollary 2.14 (2) and Remark 2.15 (3)]), Γ has full measure in E^* and hence in E .

To prove (5.4), it suffices to show that for ω_1 -almost every $\xi \in \Gamma$, we have

$$(5.5) \quad \limsup_{r \rightarrow 0} \frac{\omega_1(B(\xi, 2r))}{\omega_1(B(\xi, r))} < \infty,$$

$$(5.6) \quad \liminf_{r \rightarrow 0} \min_{i=1,2} \frac{\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i)}{r^{n+1}} > 0,$$

and

$$(5.7) \quad \lim_{r \rightarrow 0} \beta_{\omega_1, 1}(\xi, r) \frac{r^n}{\omega_1(B(\xi, r))} = 0.$$

We then use some standard measure theory to find our desired sets E_m .

The following is proven in [AMT, Lemma 5.8]. There we assume a capacity density condition, but the assumption is not used in the proof.

Lemma 5.2. *Let $\xi \in \Gamma$, $c_j \geq 0$, and $r_j \rightarrow 0$ be such that $\omega_1^j = c_j T_{\xi, r_j}[\omega_1] \rightarrow \omega_1^\infty$. Then $\omega_2^j = c_j T_{\xi, r_j}[\omega_2] \rightarrow h(\xi)\omega_1^\infty$.*

Let

$$\mathcal{F} = \{ c\mathcal{H}^n|_V : c > 0, V \text{ a } d\text{-dimensional plane containing the origin} \}.$$

It is not hard to show that \mathcal{F} has compact basis.

Lemma 5.3. *For ω_1 -a.e. $\xi \in \Gamma$,*

$$\text{Tan}(\omega_1, \xi) \cap \mathcal{F} \neq \emptyset.$$

Proof. Recall that we denote $\mathbb{B} = B(0, 1)$.

Let $\omega_1^\infty \in \text{Tan}(\omega_1, \xi)$, $c_j \geq 0$, and $r_j \rightarrow 0$ be such that $\omega_1^j = c_j T_{\xi, r_j}[\omega_1] \rightarrow \omega_1^\infty$. As $\omega_1^\infty \neq 0$, there is $R > 0$ so that $\omega_1^\infty(B(0, R)) \neq 0$. Without loss of generality, we will assume $R = 1/4$, and we can pick c_j so that

$$(5.8) \quad \omega_1^\infty\left(\frac{1}{4}\mathbb{B}\right) = 1.$$

Let $\Omega_i^j = T_{\xi, r_j}(\Omega_i)$. Let $u_1(x) = G_{\Omega_1}(x, p_1)$ on Ω_1 and $u_1(x) = 0$ on $(\Omega_1)^c$ (since we are assuming Wiener regularity, this is continuous). Set

$$u_1^j(x) = c_j u_1(xr_j + \xi) r_j^{n-1}.$$

Define u_2 and u_2^j similarly.

Without loss of generality, by passing to a subsequence we may assume that

$$(5.9) \quad \mathcal{H}^{n+1}(B(\xi, r_j) \setminus \Omega_1) \geq \frac{r_j^{n+1}}{2}.$$

Thus, for $z \in B(\xi, r_j)$,

$$\omega_{\Omega_1}^z(B(\xi, \delta^{-1}r_j)) \gtrsim \frac{\mathcal{H}^{n+1}(B(\xi, r_j) \setminus \Omega_1)}{r_j^{n+1}} \gtrsim 1.$$

Hence,

$$(5.10) \quad \omega_1(B(\xi, \delta^{-1}r_j)) \gtrsim r_j^{n-1} u_1(x) \quad \text{for all } x \in B(\xi, r_j) \cap \Omega_1,$$

and so,

$$(5.11) \quad \omega_1^j(B(0, \delta^{-1})) \gtrsim u_1^j(x) \quad \text{for all } x \in \mathbb{B} \cap \Omega_1^j,$$

By Caccioppoli's inequality for subharmonic functions and the uniform boundedness of u_1^j in \mathbb{B} , we deduce that, for $i = 1, 2$,

$$\limsup_{j \rightarrow \infty} \|\nabla u_1^j\|_{L^2(\frac{1}{2}\mathbb{B})} \lesssim \limsup_{j \rightarrow \infty} \|u_1^j\|_{L^2(\mathbb{B})} \lesssim \limsup_{j \rightarrow \infty} \omega_1^j(B(0, \delta^{-1})) \leq \omega_1^\infty(\overline{B(0, \delta^{-1})})$$

See (3.7) of [KPT] for a similar argument. By the Rellich-Kondrachov theorem, the unit ball of the Sobolev space $W^{1,2}(\frac{1}{2}\mathbb{B})$ is relatively compact in $L^2(\frac{1}{2}\mathbb{B})$, and thus there exists a subsequence of the functions u_1^j which converges *strongly* in $L^2(\frac{1}{2}\mathbb{B})$ to another function $u_1^\infty \in L^2(\frac{1}{2}\mathbb{B})$. It is easy to check that

$$\int \phi d\omega_1^j = \int \Delta \phi u_1^j dx,$$

for any C^∞ function ϕ compactly supported in $\frac{1}{2}\mathbb{B}$. Then passing to a limit, it follows that

$$(5.12) \quad \int \phi d\omega_1^\infty = \int \Delta \phi u_1^\infty dx, \quad \text{for any } \phi \in C_c^\infty(\frac{1}{2}\mathbb{B}).$$

Observe now that

$$\begin{aligned}
1 &\stackrel{(5.8)}{=} \omega_1^\infty\left(\frac{1}{4}\mathbb{B}\right) \leq \int \phi \, d\omega_1^\infty = \int_{\Omega_1} u_1^\infty \Delta \phi \, dx = \lim_j \int_{\Omega_1^j} u_1^j \Delta \phi \, dx \\
&\leq \lim_j \left(\int_{\mathbb{B} \cap \Omega_1^j \cap \{u_1^j > t\}} u_1^j \Delta \phi \, dx + \int_{\mathbb{B} \cap \Omega_1^j \cap \{u_1^j \leq t\}} u_1^j \Delta \phi \, dx \right) \\
&\leq \lim_j \inf \left(|\{x \in \mathbb{B} \cap \Omega_1^j : u_1^j > t\}| \cdot \|u_1^j\|_{L^\infty(\mathbb{B} \cap \Omega_1^j)} \|\Delta \phi\|_{L^\infty(\mathbb{B})} \right) + t \|\Delta \phi\|_{L^\infty(\mathbb{B})} \\
&\stackrel{(5.11)}{\lesssim} \lim_j \inf \left(|\{x \in \mathbb{B} \cap \Omega_1^j : u_1^j > t\}| \omega_1^\infty \left(\overline{B(0, \delta^{-1})} \right) + t \right),
\end{aligned}$$

and so, for t small enough,

$$|\mathbb{B} \cap \Omega_1^j| \geq |\{x \in \mathbb{B} \cap \Omega_1^j : u_1^j(x) > t\}| \gtrsim \omega_1^\infty \left(\overline{B(0, \delta^{-1})} \right)^{-1}.$$

In particular,

$$(5.13) \quad |B(\xi, r_j) \setminus \Omega_2| \geq |B(\xi, r_j) \cap \Omega_1| \gtrsim r_j^{n+1} \omega_1^\infty \left(\overline{B(0, \delta^{-1})} \right)^{-1}.$$

Thus, by the same arguments as earlier in proving (5.11), we have that for j large,

$$(5.14) \quad \omega_2^j(B(\xi, \delta^{-1}r_j)) \gtrsim u_2^j(x) \omega_1^\infty \left(\overline{B(\xi, \delta^{-1}r_j)} \right)^{-1}, \quad \text{for all } x \in B(\xi, r_j) \cap \Omega_2.$$

Again, we can pass to a subsequence so that u_2^j converges in $L^2(\frac{1}{2}\mathbb{B})$ to a function u_2^∞ , and it holds

$$(5.15) \quad \int \phi \, d\omega_2^\infty = \int \Delta \phi u_2^\infty \, dx.$$

Now set $u^\infty = u_1^\infty - h(\xi)^{-1}u_2^\infty$. Then by Lemma 5.2,

$$\begin{aligned}
\int u^\infty \Delta \phi &= \int u_1^\infty \Delta \phi - h(\xi)^{-1} \int u_2^\infty \Delta \phi = \int \phi \, d\omega_1^\infty - h(\xi)^{-1} \int \phi \, d\omega_2^\infty \\
&= \int \phi \, d\omega_1^\infty - h(\xi)^{-1} h(\xi) \int \phi \, d\omega_1^\infty = 0,
\end{aligned}$$

for all $\phi \in C_c^\infty(\frac{1}{2}\mathbb{B})$. Therefore, u^∞ is harmonic in $\frac{1}{2}\mathbb{B}$.

Next we claim that $u^\infty \not\equiv 0$ and that

$$(5.16) \quad \frac{1}{2}\mathbb{B} \cap \text{supp } \omega_1^\infty = \{u^\infty = 0\} \cap \frac{1}{2}\mathbb{B}.$$

First note that as $u_i^j \rightarrow u_i^\infty$ in $L^2(\frac{1}{2}\mathbb{B})$ and u_i^j have disjoint supports for all j , we know that

$$0 = \lim_{j \rightarrow \infty} \int_{\frac{1}{2}\mathbb{B}} u_1^j u_2^j \, dx = \int_{\frac{1}{2}\mathbb{B}} u_1^\infty u_2^\infty \, dx$$

and so u_1^∞ and u_2^∞ cannot be nonzero simultaneously in $\frac{1}{2}\mathbb{B}$, except in a set of zero Lebesgue measure. Since $u_1^\infty \not\equiv 0$ (by (5.12)), this implies that $u^\infty \not\equiv 0$. Another consequence is that, in $\frac{1}{2}\mathbb{B}$,

$$(5.17) \quad u_1^\infty = u^\infty \chi_{\{u^\infty > 0\}} \quad \text{and} \quad u_2^\infty = -h(\xi) u^\infty \chi_{\{u^\infty < 0\}},$$

which in particular implies that u_1^∞ and u_2^∞ are continuous in $\frac{1}{2}\mathbb{B}$, because u^∞ is harmonic there.

Observe now that for each $i = 1, 2$,

$$(5.18) \quad \text{supp } \omega_1^\infty \cap \frac{1}{2}\mathbb{B} = \partial\{u_i^\infty > 0\} \cap \frac{1}{2}\mathbb{B}.$$

This is essentially proven in [AAM, Lemma 4.7]. We omit the details. Thus, we have

$$\text{supp } \omega_1^\infty \cap \frac{1}{2}\mathbb{B} \subset \{u_1^\infty = 0\} \cap \{u_2^\infty = 0\} \cap \frac{1}{2}\mathbb{B} \subset \{u^\infty = 0\} \cap \frac{1}{2}\mathbb{B}.$$

For the converse inclusion, note that if $x \in \frac{1}{2}\mathbb{B}$ and $u^\infty(x) = 0$, then $u_1^\infty(x) = u_2^\infty(x) = 0$, since $u^\infty = u_1^\infty - h(\xi)^{-1}u_2^\infty$ and we have just shown that u_1^∞ and u_2^∞ cannot be positive simultaneously. Further, u^∞ cannot vanish identically in any ball containing x in $\frac{1}{2}\mathbb{B}$ (because it is harmonic and not identically 0), and thus either u_1^∞ or u_2^∞ must be positive in that ball. These two facts imply

$$\{u^\infty = 0\} \cap \frac{1}{2}\mathbb{B} \subset (\partial\{u_1^\infty > 0\} \cup \partial\{u_2^\infty > 0\}) \cap \frac{1}{2}\mathbb{B} = \text{supp } \omega_1^\infty \cap \frac{1}{2}\mathbb{B}.$$

This proves the claim.

In particular, $\frac{1}{2}\mathbb{B} \cap \text{supp } \omega_1^\infty$ is a smooth real analytic variety. Then, arguing as in [AMT], for example, one deduces that

$$d\omega_1^\infty|_{\frac{1}{2}\mathbb{B}} = -c_n(\nu_{\Omega_1^\infty} \cdot \nabla u_1^\infty) d\mathcal{H}^n|_{\partial^*\Omega_1^\infty \cap \frac{1}{2}\mathbb{B}},$$

where $\partial^*\Omega_1^\infty$ is the reduced boundary of $\Omega_1^\infty = \{u_1^\infty > 0\}$ and $\nu_{\Omega_1^\infty}$ is the measure theoretic outer unit normal. Hence, ω_1^∞ is absolutely continuous with respect to surface measure of $\partial\Omega_1^\infty$ in $\frac{1}{2}\mathbb{B}$. Thus, since the tangent measure at \mathcal{H}^n -almost every point of $\partial\Omega_1^\infty$ is contained in \mathcal{F} , using Lemma 4.4, we can take another tangent measure of ω_1^∞ that is in \mathcal{F} and apply Theorem 4.6 \square

The following lemma has an identical proof to that of [AMT, Lemma 5.11].

Lemma 5.4. *Let Ω_1 and Ω_2 be as above and let $\xi \in \Gamma$. If $\text{Tan}(\omega_1, \xi) \cap \mathcal{F} \neq \emptyset$, then*

$$\lim_{r \rightarrow 0} d_1(T_{\xi, r}[\omega_1], \mathcal{F}) = 0.$$

In particular, $\text{Tan}(\omega_1, \xi) \subset \mathcal{F}$.

By Theorem 4.6 and Lemma 5.4, $\text{Tan}(\omega_1, \xi) \subset \mathcal{F}$ for ω_1 a.e. $\xi \in \Gamma$. By Theorem 4.3, ω_1 and ω_2 are pointwise doubling at each such point, which proves (5.5). Also, Lemma 3.4 implies (5.6). We will now show that (5.7) holds for such a ξ .

Let $\omega_r = T_{\xi, r}[\omega_1]$. By the compactness of \mathcal{F} and the definition of d_1 , there is an n -plane V such that, if $\mu = \mathcal{H}^n|_V / F_1(\mathcal{H}^n|_V)$, then

$$(5.19) \quad \lim_{r \rightarrow 0} F_1(\omega_r / F_1(\omega_r), \mu) = 0.$$

Let ϕ be a 2-Lipschitz function which equals 1 on $\frac{1}{2}\mathbb{B}$ and 0 on \mathbb{B}^c , and set $\psi = \text{dist}(x, V)\phi$. Note that for $r < r_0/2$, (5.1) implies $F_1(\omega_r) \lesssim \omega_r(\frac{1}{2}\mathbb{B})$, and so

$$\begin{aligned} F_1(\omega_r/F_1(\omega_r), \mu) &\gtrsim F_1(\omega_r)^{-1} \int_{\mathbb{B}} \psi(x) d\omega_r^1(x) - \int_{\mathbb{B}} \psi(x) d\mu(x) \\ &\gtrsim \int_{\frac{1}{2}\mathbb{B}} \text{dist}(x, V) d\omega_r(x) - 0 \\ &= \int_{B(\xi, r/2)} \frac{\text{dist}(x, rV + \xi)}{r} d\omega_1(x) \geq \frac{(r/2)^n}{\omega_1(B(\xi, r/2))} \beta_{\omega_1, 1}(\xi, r/2). \end{aligned}$$

This and (5.19) imply (5.7).

To conclude the proof of Lemma 5.1, for $j, k \in \mathbb{N}$, set

$$\begin{aligned} E_{j,k} &= \left\{ \xi \in \Gamma : \omega_i(B(\xi, 2r)) \leq k \omega_i(B(\xi, r)), \right. \\ &\quad \mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i) \geq k^{-1} r^{n+1}, \text{ and} \\ &\quad \left. \beta_{\omega_1, 1}(\xi, r) \frac{r^n}{\omega_1(B(\xi, r))} < \varepsilon \text{ for } i \in \{1, 2\}, \quad 0 < r < 1/j \right\}. \end{aligned}$$

Then we have shown above that almost every $\xi \in \Gamma$ lies in one of these sets, and so there must be one for which $\omega_1(E_{j,k}) > 0$. Setting $F = E_{j,k}$, $r_0 = 1/j$, and $C = c^{-1} = j$ finishes the proof of Lemma 5.1.

6. RIESZ TRANSFORMS

In this section we will complete the proof of Theorem 1.1 under the additional assumption that both Ω_1 and Ω_2 are Wiener regular. So given $E \subset \partial\Omega_1 \cap \partial\Omega_2$ so that $\omega_1 \ll \omega_2 \ll \omega_1$ on E , we have to show that E contains an n -rectifiable subset F on which ω_1, ω_2 are mutually absolutely continuous with respect to \mathcal{H}^n .

Reducing E if necessary, we may assume that $\text{diam}(E) \leq \frac{1}{10} \min(\text{diam}(\Omega_1), \text{diam}(\Omega_2))$. Let \tilde{B} be some ball centered at E with radius $r(\tilde{B}) = 2 \text{diam}(E)$. We choose the poles p_i for ω_i so that $p_i \in \Omega_i \cap 2\tilde{B} \setminus \tilde{B}$. Further, by interchanging Ω_1 and Ω_2 if necessary, we may assume also that

$$\mathcal{H}^{n+1}(\tilde{B} \setminus \Omega_1) \approx \mathcal{H}^{n+1}(\tilde{B}),$$

so that, by Lemma 2.2

$$(6.1) \quad \omega_1(2\delta^{-1}\tilde{B}) = \omega_1^{p_1}(2\delta^{-1}\tilde{B}) \approx 1.$$

Given $\gamma > 0$, a Borel measure μ and a ball $B \subset \mathbb{R}^{n+1}$, we denote

$$P_{\gamma, \mu}(B) = \sum_{j \geq 0} 2^{-j\gamma} \Theta_{\mu}(2^j B),$$

where $\Theta_{\mu}(B) = \frac{\mu(B)}{r(B)^n}$.

Given $a, \gamma > 0$, we say that a ball B is a - $P_{\gamma, \mu}$ -doubling if

$$P_{\gamma, \mu}(B) \leq a \Theta_{\mu}(B).$$

Lemma 6.1. *There is $\gamma_0 \in (0, 1)$ so that the following holds. Let $\Omega \subset \mathbb{R}^{n+1}$ be any domain and ω its harmonic measure. For all $\gamma > \gamma_0$, there exists some big enough constant $a = a(\gamma, n) > 0$ such that for ω -a.e. $x \in \mathbb{R}^{n+1}$ there exists a sequence of a - $P_{\gamma, \omega}$ -doubling balls $B(x, r_i)$, with $r_i \rightarrow 0$ as $i \rightarrow \infty$.*

From now on we assume that a and γ are fixed constants such that for any domain $\Omega \subset \mathbb{R}^{n+1}$, for ω -a.e. $x \in \mathbb{R}^{n+1}$ there exists a sequence of a - $P_{\gamma, \omega}$ -doubling balls $B(x, r_i)$, with $r_i \rightarrow 0$ as $i \rightarrow \infty$.

Recall that the harmonic measures ω_1 and ω_2 are mutually absolutely continuous on $E \subset \partial\Omega_1 \cap \partial\Omega_2$, and that h denotes the density function $h(\xi) = \frac{d\omega_2}{d\omega_1}(\xi)$ and that we assume that Ω_1, Ω_2 are Wiener regular.

Let E_m be one of the sets from Lemma 5.1 and fix $m \geq 1$ so that $\omega_1(E_m) > 0$.

Lemma 6.2. *Let $m \geq 1$ and $\delta > 0$. For ω_1 -a.e. $x \in E_m$, there is $r_x > 0$ so that for any a - P_{γ, ω_1} -doubling ball $B(x, r)$ with radius $r \leq r_x$ there exists a subset $G_m(x, r) \subset E_m \cap B(x, r)$ such that*

$$(6.2) \quad \frac{\omega_1(B(z, t))}{t^n} \lesssim \frac{\omega_1(B(x, r))}{r^n} \quad \text{for all } z \in G_m(x, r), 0 < t \leq 2r,$$

and so that $\omega_1(B(x, r) \setminus G_m(x, r)) \leq \delta \omega_1(B(x, r))$.

The proof is almost the same as the one of the analogous Lemma 6.2 from [AMT]. The only change is that we cannot rely on Lemma 4.11 from [AMT], and instead we use the fact that, by Lemmas 3.2 and 3.5, given $\xi \in \partial\Omega_1 \cap \partial\Omega_2$ and $0 < s < r$, with r small enough, if $\omega_i(B(\xi, 4r)) \leq C \omega_i(B(\xi, \delta_0 r))$ for $i = 1, 2$ (which is guaranteed by Lemma 5.1), then we have

$$\gamma(\xi, s)^{\frac{1}{2}} \leq \gamma(\xi, r)^{\frac{1}{2}} \lesssim \frac{\omega_1(B(\xi, 8\delta_0^{-1}r))}{r^n} \frac{\omega_2(B(\xi, 8\delta_0^{-1}r))}{r^n}.$$

Given $m \geq 1$ and $\delta > 0$, we denote by $\tilde{E}_{m, \delta}$ the subset of points $x \in E_m$ for which there exists $r_x > 0$ as in Lemma 6.2, so that $\omega_1(E_m \setminus \tilde{E}_{m, \delta}) = 0$.

Lemma 6.3. *Let $m \geq 1$ and $\delta > 0$. Let $x_0 \in \tilde{E}_{m, \delta}$ and*

$$0 < r_0 \leq \min(r_{x_0}, 1/m, c_1 \text{dist}(p_1, \partial\Omega_1)),$$

for some $c_1 > 0$ small enough (recall that ω_i is the harmonic measure for Ω_i with pole at p_i). Suppose that the ball $B_0 = B(x_0, r_0)$ is a - P_{γ, ω_1} -doubling. Then for all $x \in G_m(x_0, r_0)$ it holds that

$$(6.3) \quad \mathcal{R}_*(\chi_{2B_0}\omega_1)(x) \lesssim \Theta_{\omega_1}(B_0).$$

In the proof of the analogous lemma in [AMT] we used the fact that, for small radii, the β_∞ coefficients of the boundary for balls centered at E are small ω_1 -a.e. in the case that Ω_1 and Ω_2 satisfy the CDC. This is no longer true (as far as we know), and so the arguments below are somewhat different (in fact, they are inspired by the estimates in the Key Lemma 4.3 from [AHM³TV]).

Proof. To estimate $|\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)|$ for $x \in G_m(x_0, r_0)$ we may assume that $r \leq r_0/4$ because $|\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)| = 0$ if $r \geq 4r_0$ and (6.3) is trivial in the case $r_0/4 < r < 4r_0$.

So we take $x \in G_m(x_0, r_0)$ and $0 < r \leq r_0/4$. Note that

$$\begin{aligned} |\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)| &= |\mathcal{R}_r\omega_1(x) - \mathcal{R}_r(\chi_{(2B_0)^c}\omega_1)(x)| \\ &\leq |\mathcal{R}_r\omega_1(x) - \mathcal{R}_{r_0/4}\omega_1(x)| + |\mathcal{R}_{r_0/4}\omega_1(x) - \mathcal{R}_r(\chi_{(2B_0)^c}\omega_1)(x)|. \end{aligned}$$

It is immediate to check that the last term is bounded above by $C\Theta_{\omega_1}(2B_0)$, and thus

$$(6.4) \quad |\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)| \leq |\mathcal{R}_r\omega_1(x) - \mathcal{R}_{r_0/4}\omega_1(x)| + C\Theta_{\omega_1}(B_0).$$

Let $\varphi : \mathbb{R}^{n+1} \rightarrow [0, 1]$ be a radial C^∞ function which vanishes on $B(0, 1)$ and equals 1 on $\mathbb{R}^{n+1} \setminus B(0, 2)$, and for $\varepsilon > 0$ and $z \in \mathbb{R}^{n+1}$ denote $\varphi_\varepsilon(z) = \varphi(\frac{z}{\varepsilon})$ and $\psi_\varepsilon = 1 - \varphi_\varepsilon$. We set

$$\tilde{\mathcal{R}}_\varepsilon\omega_1(z) = \int K(z - y) \varphi_\varepsilon(z - y) d\omega_1(y),$$

where $K(\cdot)$ is the kernel of the n -dimensional Riesz transform. Note that

$$|\tilde{\mathcal{R}}_r\omega_1(x) - \mathcal{R}_r\omega_1(x)| \lesssim \Theta_{\omega_1}(B(x, 2r)).$$

Therefore, by (6.4) and (6.2),

$$(6.5) \quad |\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)| \leq |\tilde{\mathcal{R}}_r\omega_1(x) - \tilde{\mathcal{R}}_{r_0/4}\omega_1(x)| + C\Theta_{\omega_1}(B_0).$$

To estimate the first term on the right hand side of the inequality above, for a fixed $x \in G_m(x_0, r_0)$ and $z \in \mathbb{R}^{n+1} \setminus [\text{supp}(\varphi_r(x - \cdot)\omega_1) \cup \{p_1\}]$, consider the function

$$v_r(z) = \mathcal{E}(z - p_1) - \int \mathcal{E}(z - y) \varphi_r(x - y) d\omega_1(y),$$

so that, by Remark 3.2 from [AHM³TV],

$$(6.6) \quad u_1(z) = G_{\Omega_1}(z, p_1) = v_r(z) - \int \mathcal{E}(z - y) \psi_r(x - y) d\omega_1(y) \quad \text{for } m\text{-a.e. } z \in \mathbb{R}^{n+1}.$$

Since the kernel of the Riesz transform is

$$(6.7) \quad K(x) = c_n \nabla \mathcal{E}(x),$$

for a suitable absolute constant c_n , we have

$$\nabla v_r(z) = c_n K(z - p) - c_n \mathcal{R}(\varphi_r(\cdot - x)\omega_1)(z).$$

In the particular case $z = x$ we get

$$\nabla v_r(x) = c_n K(x - p) - c_n \tilde{\mathcal{R}}_r\omega_1(x).$$

Using this identity also for $r_0/4$ instead of r , we obtain

$$(6.8) \quad |\tilde{\mathcal{R}}_r\omega_1(x) - \tilde{\mathcal{R}}_{r_0/4}\omega_1(x)| \approx |\nabla v_r(x) - \nabla v_{r_0/4}(x)| \leq |\nabla v_r(x)| + |\nabla v_{r_0/4}(x)|$$

Since v_r is harmonic in $\mathbb{R}^{n+1} \setminus [\text{supp}(\varphi_r(x - \cdot)\omega_1) \cup \{p_1\}]$ (and so in $B(x, r)$), we have

$$(6.9) \quad |\nabla v_r(x)| \lesssim \frac{1}{r} \int_{B(x, r)} |v_r(z)| dm(z).$$

From the identity (6.6) we deduce that

$$(6.10) \quad \begin{aligned} |\nabla v_r(x)| &\lesssim \frac{1}{r} \int_{B(x, r)} u_1(z) dm(z) + \frac{1}{r} \int_{B(x, r)} \int \mathcal{E}(z - y) \psi_r(x - y) d\omega_1(y) dm(z) \\ &=: I + II. \end{aligned}$$

To estimate the term II we use Fubini and the fact that $\text{supp } \varphi_r \subset B(x, 2r)$:

$$II \lesssim \frac{1}{r^{n+2}} \int_{y \in B(x, 2r)} \int_{z \in B(x, r)} \frac{1}{|z - y|^{n-1}} dm(z) d\omega_1(y) \lesssim \frac{\omega_1(B(x, 2r))}{r^n} \lesssim \Theta_{\omega_1}(B_0).$$

We intend to show now that $I \lesssim \Theta_{\omega_1}(B_0)$. Clearly it is enough to show that

$$(6.11) \quad \frac{1}{r} |u_1(y)| \lesssim \Theta_{\omega_1}(B_0) \quad \text{for all } y \in B(x, r) \cap \Omega.$$

To prove this, observe that by Lemma 2.1 (with $B = B(x, r)$, $a = 2\delta_0^{-1}$), for all $y \in B(x, r) \cap \Omega$, we have

$$\omega_1(B(x, 2\delta_0^{-1}r)) \gtrsim \inf_{z \in B(x, 2r) \cap \Omega} \omega^z(B(x, 2\delta_0^{-1}r)) r^{n-1} |u_1(y)|.$$

On the other hand, by Lemma 2.2, for any $z \in B(x, 2r) \cap \Omega_1$,

$$\omega_1^z(B(x, 2\delta_0^{-1}r)) \gtrsim \frac{\mathcal{H}^{n+1}(B(x, 2r) \setminus \Omega_1)}{r^{n+1}} \gtrsim 1.$$

Therefore, $\omega_1(B(x, 2\delta_0^{-1}r)) \gtrsim r^{n-1} |u_1(y)|$. Then,

$$\frac{1}{r} |u_1(y)| \lesssim \Theta_{\omega_1}(B(x, 2\delta_0^{-1}r)) \lesssim P_{\gamma, \omega_1}(B_0) \lesssim \Theta_{\omega_1}(B_0),$$

which proves (6.11).

By the estimates obtained above for the terms I , II for r and $r_0/4$, we derive

$$|\nabla v_r(x)| + |\nabla v_{r_0/4}(x)| \lesssim \Theta_{\omega_1}(B_0).$$

Hence, by (6.5) and (6.8), we infer that

$$|\mathcal{R}_r(\chi_{2B_0}\omega_1)(x)| \lesssim \Theta_{\omega_1}(B_0),$$

as wished. \square

Let $m \geq 1$, $\delta > 0$, and $x_0 \in \tilde{E}_{m, \delta}$, and denote

$$G_m^{zd}(x_0, r_0) = \{x \in G_m(x_0, r_0) : \lim_{r \rightarrow 0} \Theta_{\omega_1}(B(x, r)) = 0\},$$

and

$$G_m^{pd}(x_0, r_0) = \{x \in G_m(x_0, r_0) : \limsup_{r \rightarrow 0} \Theta_{\omega_1}(B(x, r)) > 0\}.$$

The notation “ zd ” stands for “zero density”, and “ pd ” stands for “positive density”.

The proof of the next lemma is the same as the one of the analogous lemma in [AMT]. This is an easy consequence of the main result from [AHM³TV], which in turn relies on [NTV1] and [NTV2].

Lemma 6.4. *Let $m \geq 1$ and $\delta > 0$. Let $x_0 \in \tilde{E}_{m, \delta}$ and*

$$0 < r_0 \leq \min(r_{x_0}, 1/m, c_1 \text{dist}(p_1, \partial\Omega_1)),$$

for some $c_1 > 0$ small enough. Suppose that the ball $B_0 = B(x_0, r_0)$ is a - P_{γ, ω_1} -doubling.

Then there is an n -rectifiable set $F(x_0, r_0) \subset G_m^{pd}(x_0, r_0)$ such that

$$\omega_1(G_m^{pd}(x_0, r_0) \setminus F(x_0, r_0)) = 0$$

and so that $\omega_1|_{F(x_0, r_0)}$ and $\mathcal{H}^n|_{F(x_0, r_0)}$ are mutually absolutely continuous.

Lemma 6.5. *Let $m \geq 1$ and $\delta > 0$. Let $x_0 \in \tilde{E}_{m,\delta}$ and*

$$0 < r_0 \leq \min(r_{x_0}, 1/m, c_1 \text{dist}(p_1, \partial\Omega_1)),$$

for some $c_1 > 0$ small enough. Suppose that the ball $B_0 = B(x_0, r_0)$ is a - P_{γ, ω_1} -doubling, then

$$(6.12) \quad \int_{G_m^{zd}(x_0, r_0)} |\mathcal{R}\omega_1(x) - m_{\omega_1, G_m^{zd}(x_0, r_0)}(\mathcal{R}\omega_1)|^2 d\omega_1(x) \lesssim \left(\frac{r_0}{|x_0 - p_1|} \right)^{2-2\gamma} \Theta_{\omega_1}(B_0)^2 \omega_1(B_0).$$

Proof. We claim that for ω_1 -a.e. $x \in G_m^{zd}(x_0, r_0)$,

$$(6.13) \quad \mathcal{R}\omega_1(x) = K(x - p_1).$$

Indeed, consider a sequence $r_j \rightarrow 0$ so that $B_j = B(x, r_j)$ is a - P_{ω_1} -doubling for every j . Then if $\tilde{\mathcal{R}}_r$ is as in the proof of Lemma 6.1, we have that $\tilde{\mathcal{R}}_{r_j}\omega_1(x) - K(x - p_1) = c_n \nabla v_{r_j}(x)$. From the exact same estimates we can prove that

$$|\nabla v_{r_j}(x)| \lesssim \frac{\omega_1(B(x, 2\delta_0^{-1}r_j))}{r_j^n} \lesssim \frac{\omega_1(B(x, r_j))}{r_j^n},$$

where in the last inequality we used that $B_j = B(x, r_j)$ is a - P_{ω_1} -doubling. Therefore, taking $j \rightarrow \infty$ and since $\lim_{r \rightarrow 0} \Theta_{\omega_1}(B(x, r)) = 0$, we infer that $\mathcal{R}\omega_1(x) = K(x - p_1)$. Since the distance between the center \tilde{z} of \tilde{B} (this is the ball introduced at the beginning of this section) and x_0 is at most $\text{diam}(E)$ and $|\tilde{z} - p_1| \geq 2 \text{diam}(E)$, it follows easily that $\delta_0^{-1}\tilde{B} \subset B(x_0, 10\delta_0^{-1}|x_0 - p_1|)$, and then from (6.1) we infer that

$$\omega_1(B(x_0, 10\delta_0^{-1}|x_0 - p_1|)) \approx 1.$$

Then we obtain

$$\begin{aligned} |\mathcal{R}\omega_1(x) - m_{\omega_1, G_m^{zd}(x_0, r_0)}(\mathcal{R}\omega_1)| &\leq \sup_{y \in G_m^{zd}(x_0, r_0)} |K(x - p_1) - K(y - p_1)| \\ &\lesssim \frac{r_0}{|x_0 - p_1|^{n+1}} \lesssim \frac{r_0}{|x_0 - p_1|} \frac{\omega_1(B(x_0, 10\delta_0^{-1}|x_0 - p_1|))}{|x_0 - p_1|^n} \\ &= \Theta_{\omega_1}(B(x_0, 10\delta_0^{-1}|x_0 - p_1|)) \left(\frac{r_0}{|x_0 - p_1|} \right)^\gamma \left(\frac{r_0}{|x_0 - p_1|} \right)^{1-\gamma} \\ &\lesssim P_{\gamma, \omega_1}(B_0) \left(\frac{r_0}{|x_0 - p_1|} \right)^{1-\gamma} \\ &\lesssim \Theta_{\omega_1}(B_0) \left(\frac{r_0}{|x_0 - p_1|} \right)^{1-\gamma}. \end{aligned}$$

where in the last inequality we used the fact that B_0 is a - P_{γ, ω_1} -doubling. The conclusion of the lemma readily follows. \square

To finish the proof of Theorem 1.1 we proceed as in [AMT]: by combining the preceding lemma with the main result from [GT], it follows easily that $\omega_1(G_m^{zd}(x_0, r_0)) = 0$. From this fact and Lemma 6.4, one deduces Theorem 1.1. The precise arguments are the same as the ones in the end of Section 6 of [AMT].

We have now proven Theorem 1.1 for regular domains. We now apply Lemma 2.3 to obtain the general case. This finishes the proof of Theorem 1.1.

REFERENCES

- [AAM] M. Akman, J. Azzam, and M. Mouroglou, *Absolute continuity of harmonic measure for domains with lower regular boundaries*. *arXiv preprint arXiv:1605.07291* (2016).
- [ACF] H.W. Alt, L.A. Caffarelli, and A. Friedman, *Variational problems with two phases and their free boundaries*. *Trans. Amer. Math. Soc.* **282** (1984), no. 2, 431–461. MR 732100 (85h:49014)
- [AG] D.H. Armitage and S.J. Gardiner, *Classical potential theory*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2001.
- [AHM³TV] J. Azzam, S. Hofmann, J.M. Martell, S. Mayboroda, M. Mouroglou, X. Tolsa, and A. Volberg. *Rectifiability of harmonic measure*. *Geom. Funct. Anal. (GAFA)*, 26(3) (2016), 703-728.
- [AMT] J. Azzam, M. Mouroglou and X. Tolsa. *Mutual absolute continuity of interior and exterior harmonic measure implies rectifiability*. Preprint (2015). To appear in *Comm. Pure Appl. Math.*
- [Bi1] C. J. Bishop, *Some questions concerning harmonic measure*, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), IMA Vol. Math. Appl., vol. 42, Springer, New York, 1992, pp. 89–97. MR 1155854 (93f:30023)
- [Bi2] C. J. Bishop, *A characterization of Poissonian domains*. *Ark. Mat.* **29(1)** (2006), 1–24.
- [CS] L. Caffarelli and S. Salsa. *A geometric approach to free boundary problems*. Graduate Texts in Math. 64. Amr. Math. Soc. (2005).
- [ENV] V. Eiderman, F. Nazarov and A. Volberg. *The s -Riesz transform of an s -dimensional measure in \mathbb{R}^2 is unbounded for $1 < s < 2$* . *J. Anal. Math.* (2014) 122: 1.
- [GT] D. Girela-Sarrión and X. Tolsa, *Quantitative estimates for the Riesz transform and rectifiability for general Radon measures*. Preprint arXiv:1601.08079 (2015).
- [HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.
- [H] L.L. Helms, *Potential theory*, Universitext, Springer, London, 2014, 2nd ed.
- [HMMTV] S. Hofmann, J.M. Martell, S. Mayboroda, X. Tolsa, and A. Volberg. *Absolute continuity between the surface measure and harmonic measure implies rectifiability*. Preprint arXiv:1507.04409 (2015).
- [KPT] C.E. Kenig, D. Preiss, and T. Toro, *Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions*, *J. Amer. Math. Soc.* **22** (2009), no. 3, 771–796. MR 2505300 (2010h:28005)
- [Ma] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.
- [NTV1] F. Nazarov, X. Tolsa and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1*, *Acta Math.* **213** (2014), no. 2, 237–321. MR 3286036
- [NTV2] F. Nazarov, X. Tolsa and A. Volberg. *The Riesz transform Lipschitz harmonic functions*. *Publ. Mat.* 58 (2014), 517–532.
- [Pr] D. Preiss, *Geometry of measures in \mathbb{R}^n : distribution, rectifiability, and densities*, *Ann. of Math.* 125 (1987), 537–643.
- [Ts] B. Tsirelson. *Triple points: from non-Brownian filtrations to harmonic measures*. *Geom. Funct. Anal. (GAFA)*, Vol. 7 (1997) 1096-1142.
- [TV] X. Tolsa and A. Volberg. *On Tsirelson’s theorem about triple points for harmonic measure*. Preprint arXiv:1608.04022 (2016).

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JCMB, KINGS BUILDINGS, MAYFIELD ROAD, EDINBURGH, EH9 3JZ, SCOTLAND.

E-mail address: j.azzam "at" ed.ac.uk

MIHALIS MOURGOGLOU, DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, EDIFICI C FACULTAT DE CIÈNCIES, 08193 BELLATERRA (BARCELONA), CATALONIA

E-mail address: mourgoglou@mat.uab.cat

XAVIER TOLSA, ICREA, PASSEIG LLUS COMPANYS 23 08010 BARCELONA, CATALONIA, AND, DEPARTAMENT DE MATEMÀTIQUES AND BGS MATH, UNIVERSITAT AUTÒNOMA DE BARCELONA, EDIFICI C FACULTAT DE CIÈNCIES, 08193 BELLATERRA (BARCELONA), CATALONIA

E-mail address: xtolsa@mat.uab.cat

ALEXANDER VOLBERG, DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, USA

E-mail address: volberg@math.msu.edu